Estimation of the Stochastic Volatility of a Diffusion Process I. Comparison of Haar basis Estimator and some Kernel Estimators

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Estimation of the Stochastic Volatility of a Diffusion Process I. Comparison of Haar basis Estimator and Kernel Estimators.

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Abstract: Let (X_t) be a stochastic process satisfying

$$dX_t = b(t, X_t)\,dt + \theta(t)\,dW_t$$

with a stochastic volatility $\theta(t)$ (thus less regular than $\mathcal{C}^1$). We have a discretized observation at discrete times $t_i = i\Delta$ for $i = 1, \ldots, N$. We want to estimate $\theta(t)$. We compare three families of non-parametric estimators: Wavelet Estimator in the Haar basis, Moving Average Estimator and Centered Moving Average Estimator (CMAE). We emphasis the dependence of the estimators on the size of the observation window. This is a new point of view. We prove the punctual convergence of the three estimators at the same rate. Then, we study Mean Integrated Square Error (MISE) as a function of the window size, we show it is smaller for Centred Moving Average Estimator (CMAE) than for Haar Basis Estimator in most circumstances. We prove a Central Limit Theorem for Integrated Square Error (ISE) in the deterministic case. We conclude by numerical simulations which illustrate our theorical results. AMS Classifications. 62M 05, 60G 35.

Key-words: Stochastic Volatility, Diffusion Process, Non-parametric Estimation, Centred Moving Average Estimator, Wavelet Estimator, Mean Integrated Square Error, Stochastic numerical Algorithms.

(Résumé : tsvp)

*Ce travail s'inscrit dans le cadre d'une collaboration avec D. Talay et son projet OMEGA, R.G. Avesani (Université de Brescia) et L. Tubaro (Université de Trente). Le problème d'identification de volatilités aléatoires constantes par morceaux dans des modèles financiers a été posé par R.G. Avesani. Les simulations numériques ont été faites par B. Iooss à l'INRIA Sophia-Antipolis. Je remercie également V.Genon-Catalot pour de très utiles discussions et conseils sur ces sujets.
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Estimation de la volatilité stochastique d'un processus de diffusion I. Comparaison d'estimateurs d'ondelette et d'estimateurs à noyau.

Résumé : Soit (X_t) un processus stochastique solution de l'EDS

$$dX_t = b(t, X_t)\, dt + \theta(t)\, dW_t\ .$$

On dispose de l'observation d'une trajectoire X_t à des instants discrets $t_i = i\Delta$ avec $i = 1, \ldots, N$. On veut estimer le coefficient de diffusion $\theta(t)$ (appelé volatilité dans la littérature financière) dans le cas où celui-ci serait stochastique (processus à sauts ou diffusion, par exemple). On compare trois familles d'estimateurs non-paramètriques : un estimateur d'ondelette par projection dans la base de Haar et deux estimateurs à noyau qui, de fait, correspondent à une moyenne mobile, pour le premier et à une moyenne mobile recentrée, pour le second.

Nous explicitons la dépendance des estimateurs par rapport à la taille de la fenêtre (i.e le nombre d'observations prises en compte). Ceci est nouveau.

On montre que les trois estimateurs convergent ponctuellement à la même vitesse. On étudie ensuite l'Erreur Quadratique Moyenne Intégrée (MISE) comme fonction de la fenêtre et on montre qu'elle est, le plus souvent, plus petite pour l'estimateur de la moyenne mobile recentrée que pour l'estimateur dans la base de Haar. Dès qu'il y a au moins un saut de volatilité, l'erreur quadratique moyenne intégrée est une fonction oscillante de la fenêtre pour l'estimateur d'ondelette. Par contre, ce phénomène n'apparait pas pour l'estimateur de la moyenne mobile recentrée, qui est donc plus robuste.

On démontre aussi dans le cas d'une volatilité déterministe, un Théorème Central Limite pour la différence entre l'Erreur Quadratique Intégrée (ISE) et l'Erreur Quadratique Moyenne Intégrée .

Finalement, sont effectuées des simulations numériques qui confirment pleinement les résultats théoriques.

AMS Classifications : 62M 05, 60G 35.

Mots-clé : Volatilité stochastique, Processus de diffusion, Estimation non-paramétrique, Erreur Quadratique Moyenne Intégrée, Algorithmes Numériques stochatiques

1 Introduction.

We consider a stochastic process satisfying the following Stochastic Differential Equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \qquad X(0) = x_0 \tag{1}$$

We have a discretized observation of the process $(X_t, t \in [O, T])$ at the discrete times $t_i = \ i\ \Delta$ for $i = 1, \dots, N$ and we want to estimate the diffusion coefficient $\sigma(t, X)$. The parametric case $\sigma(t, X) = \theta f(X)$ is studied in [9] and by V.Genon-Catalot & J.Jacod [16] in a general framework. The non-parametric case for an autonomous diffusion coefficent (i.e. $\sigma(t, x) = \sigma(x)$) has been treated by D.Florens-Zmirou [12] for one dimensional processes and by P.Brugière [4] and [5] for multidimensional processes. However, this Markovian assumption which implies that the volatility only depends on the process seems restrictive. E.Fournié try to verify the Cox-Ingersoll-Ross model [7] with constant coefficients on real financial data (from the french interest rates) and he concluded that, at least the diffusion coefficient is time varying [13, p.34]. *In financial literature, the diffusion coefficient is called volatility, we follow this notation* in all the sequel. D.Florens-Zmirou [11] give a non-parametric estimator for a time varying volatility $\sigma(x, t) = \theta(t)h(x)$ and in [15] V.Genon-Catalot, C.Laredo & D.Picard introduced Estimation by Wavelet methods. Both [12] and [15] consider a time varying coefficient $\theta(t)$ which is a $\mathcal{C}^1$ function of time and deterministic. However, for the financial applications it seems more natural to consider a stochastic coefficient $\theta(t)$ (see [1]), in this case the coefficient $\theta(t)$ would be less regular than $\mathcal{C}^1$: we will consider either a Hölder continuous function either a piece-wise constant function. The same description of stochastic volatility is also considered in [10].

The first aim of this paper is to study the estimation of stochastic volatility, thus less regular than $\mathcal{C}^1$ and to compare Wavelet Estimator and kernel estimators. In this case the Kernel estimator corresponds to a Moving Average (except on the boundaries) and for a symmetric flat kernel to a Centred Moving Average (called CMAE). Surprisingly, this *"oldfashioned"* Estimator improved by centering turn to be the best in most circumstances, as soon as there is a volatility jump.

The second aim is to numerically compute the differents estimators. We meet a new question: **What is the good choice for the window ?** In [15] and [27] an optimal choice is proposed: $j(n) = n/(2m+1)$ with $\Delta = 2^{-n}$ and $h_n = 2^{-j(n)}$. This holds asymptotically. In fact, we deduce from [27, Prop.1, p.785] that $j_{opt}(n) = n/(2m + 1) + \log_2(2mD/C)$, where C and D are fixed constants depending on the norms

$L^2(0,T;\gamma(t)dt)$ of $\theta^2(.)$ and its derivative of order m. For example, $n=15$, $m=1$ and $\Delta = 1/32800$ lead to $n/(2m+1)=5$ and $\log_2(2mD/C)$ could be of order 1 or 2 (in usual cases). Moreover, the constants C and D, which depend on the true value of the function $\theta(.)$, could not be precisely estimated nor disregarded behind $n/(2m+1)$.

In financial applications, the sampling interval Δ is given from the data, it is a small parameter but not asymptotically small (see also Bibby & Sorensen [2]). Therefore, we have to arbitrarily choose the size of the window. **So we decide to let the size of the window be a free parameter of the differents families of Estimators** and we change all the notations to emphasis dependence of Estimators on the window (denoted A) and the sampling interval Δ. This attention to the window size is new and seems to be a fruitful idea which will be developed in another paper to build estimators of volatility jump times (e.g. for a jump process).

Let us precise some implications of our assumption that the sampling procedure is given by the data. It would not allow us to make adaptive or optimal sampling (as in [17] or [20] for example), nor to refine the observations . This sould be considered as an opposite framework. We postulate that we are not allowed to get extra-quotations between two successive times t_i and t_{i+1}.

In this paper, we compare the different Estimators to find the best and the more robust.

Two points of view are mixed : an asymptotic point of view and an exact one (A is a free parameter and Δ is a small parameter). In the first section, we describe the model an give some financial examples. In the second section, we describe the three Estimators (depending on A and Δ) and prove they are consistent, asymptotically normal and have the same rate of convergence at each point (exempt jump times) of $[0,T]$; this is an asymtotic point of view corresponding to the asymptotic $A \to +\infty$ and $A\Delta \to 0$. In the third section, we compare functional errors: Mean Integrated Square Error (MISE), we give expression of Mean Integrated Square Error as sum of exact terms (depending on A) and smaller terms ; this is an exact point of view with Δ considered as a small parameter. When $\theta(.)$ has isolated jumps (at most one in each window), we get simpler expression of MISE and we can precisely explain why Centred Moving Average Estimator is better and more robust than Haar basis Estimator. Numerical computations show it is always true when there is at least one volatility jump. In the fourth section, we prove Central Limit Theorem for ISE in the deterministic case (again, it is an asymptotical point of view, but for a different asymptotic $\Delta \to 0$). In the last section, we conclude by numerical comparison of MISE and ISE for different functions $\theta(.)$ and numerical simulations of Estimators which plainly confirm our theorical study (again, this is an exact point of view).

Applications to real data (e.g. Future of Italian Bunds) are treated in [1].

Remark : This is off-line estimation in every case.

2 Description of the model, Assumptions.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a stochastic basis and (W_t) a one dimensional Wiener process adapted to $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We consider a stochastic process statisfying the following Stochastic Differential Equation :

$$dX_t = b(t, X_t)dt + \theta(t)\, h(X_t)\, dW_t \tag{2}$$

where the function $h(.)$ is assumed to be known, the volatility coefficient $\theta(.)$ is an unknown function of time and has to be correctly estimated, the drift coefficient $b(t,x)$ may be unknown. We observe one sampling path of the process $(X_t, t \in [0,T])$ at the discrete times $t_i = i\,\Delta$ for $i = 1, \ldots, N$. The sampling interval Δ is small in comparison of T, we denote $N := T\,\Delta^{-1}$ and assume it is an integer.

We will use the following assumption:

(A0) $\theta(t)$ *is adapted to the filtration* $(\mathcal{F}_t)$, $b(t,.)$ *is a non-anticipative map,* $b \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R})$ *and* $\exists L_T > 0$ *such that* $\forall t \in [0,T]$, $\mathbb{E}\theta^4(t) \le L_T$ *and* $\mathbb{E}\theta^8(t) \le L_T$

Moreover, we want to consider either a jump process either a diffusion process, respectively corresponding to the following assumptions:

(A1) $\theta(.) = \sum_{\rho=0}^{f} \theta_\rho\, \mathbf{1}_{[\mathbf{t}_\rho, \mathbf{t}_{\rho+1})}(.)$ *where* t_ρ *are the volatility jump times.*

(A2) $\exists m > 0$ *such that* $\theta^2(.)$ *is almost surely Hölder continuous of order* m *with a constant* $K(\omega)$ *and* $\mathbb{E}K(\omega)^2 < +\infty$.

If we assume that the volatility jump times correspond to the sampling times $t_i = i\Delta$, we have

(A1') $\theta(.) = \sum_{i=0}^{N} \theta_i\, \mathbf{1}_{[\mathbf{t_i}, \mathbf{t_{i+1}})}(.)$ *we denote* $\delta\theta_i^2 = \theta_{i+1}^2 - \theta_i^2$.

and if moreover there is at most one change time in each window we get (A3).

(A3) (A1) and (A1') are satisfied and $\inf_{\rho=0,\dots,f} \mid t_{\rho+1} - t_\rho \mid \geq A\Delta$.

Remark : If $\theta(t)$ satisfies a S.D.E. then (A2) is fulfilled, see e.g. [26, th.2.1, p.25].

We need to control $\int_{t_i}^{t_{i+1}} b^4(s, X_s)\, ds$, so we will use :

(B1) $\exists K_T > 0,\ \forall t \in [0,T],\ \mathbb{E} b(t, X_t)^4 \leq K_T$

In all the sequel we work on the simplified model:

$$dX_t = b_1(t, X_t)dt + \theta(t)\, dW_t \tag{3}$$

Under natural assumptions, the model (2) becomes (3) after the following change of variable:

Proposition 2.1 *Assume there exist a domain* $D \subseteq \mathbb{R}$ *such that* $h \in \mathcal{C}^1(D, \mathbb{R}^{+*})$, $h^{-1} \in L^1_{loc}(D)$ *and for* (X_t) *solution of (2) satisfying* $\mathbb{P}(X_t \in D, \forall t \in [0,T]) = 1$. *Let* $H(x) = \int h^{-1}(\xi) d\xi$. *Then* $Y_t = H(X_t)$ *satisfies the S.D.E. (3) with* $b_1(t,x) = h^{-1}(x)\ a(t,x) - \frac{1}{2} h'(x)\theta^2(t)$.

Proof: It follows directly from Itô Formula ■

Financial Applications:

Example 1:
Let S_t be an asset satisfying the following S.D.E.

$$dS_t = S_t \left[a(t, S_t)\, dt + \theta(t)\, dW_t \right] \tag{4}$$

Then $X_t = \ln(S_t)$ satisfies (3) with $b(t,x) = a(t, \exp(x)) - \frac{1}{2}\theta^2(t)$ and $b(t,x)$ fulfills the assumption $(B1)$. This model is also considered in [10, p.5].

Example 2:
Let X_t be an interest rate satisfying a stochastic Cox-Ingersoll-Ross Stochastic Differential Equation with time varying coeffients

$$dX_t = c(t)(\alpha(t) - X_t)\, dt + \theta(t)\, (X_t)^{1/2}\, dW_t \tag{5}$$

where $c(t)$, $\alpha(t)$ and $\theta(t)$ are $\mathcal{F}_t$ adapted. If $\mathbb{P}(X_t > 0, \forall t \in [0,T]) = 1$, then $Y_t = 2(X_t)^{1/2}$ satisfies (2) with $b(t,x) = 2x^{-1} \left[c(t)\ \alpha(t) - \frac{1}{4}\theta^2(t) \right] - \frac{1}{2}\alpha(t) x$. For stochastic coefficients $c(t), \alpha(t), \theta(t)$, positiveness of the solution X_t of (5) is proved in [1].

Example 3:

We still consider the Cox-Ingersoll-Ross S.D.E. (5), but we directly estimate $\sigma(t) = \theta(t)h(X_t)$. Obviously, $\sigma(t)$ is a stochastic process and satisfies (A2) when $\theta(t)$ satisfies (A2). We denote $\hat{\theta}_2(t) = \hat{\theta}_{CMAE}(t)$ and $\hat{\theta}_3(t) = h^{-1}(X_t)\hat{\sigma}_{CMAE}(t)$. The numerical simulations made with a known function $\theta(.)$ give quite the same results for $\hat{\theta}_2(t)$ and $\hat{\theta}_3(t)$ and almost same Integrated Square Error (for e.g. $ISE_2 = 5,9\%$ and $ISE_3 = 6\%$). But, if the function $h(.)$ is not exactly known, we still obtain a good estimation $\hat{\theta}_3(t)$ of $\theta(t)$. So, we believe that this direct estimation of $\sigma(t)$ is more robust with respect to the model than the previous one.

3 Estimation of the Volatility.

3.1 Description of the Estimators.

Since the size of the window appears in numerical applications as a free parameter to be arbitrarily choosen, we give a description of the Estimators depending explicitely on the window A (number of observations tooken into account to estimate $\theta(t)$). We think it is the good parameter to consider.

Moving Average Estimator:

$$MAE_{A,\Delta}(t) = \sum_{j=0}^{N-1} \left\{ A^{-1} \sum_{i=1}^{A-1} \Delta^{-1}(X_{t_{j-i+1}} - X_{t_{j-i}})^2 \right\} \mathbf{1}_{[\mathbf{t_j},\mathbf{t_{j+1}})}(\mathbf{t}) \tag{6}$$

Centred Moving Average Estimator:

$$CMAE_{A,\Delta}(t) = \sum_{j=0}^{N-1} \left\{ A^{-1} \sum_{i=-\frac{A}{2}}^{\frac{A}{2}-1} \Delta^{-1}(X_{t_{j-i+1}} - X_{t_{j-i}})^2 \right\} \mathbf{1}_{[\mathbf{t_j},\mathbf{t_{j+1}})}(\mathbf{t}) \tag{7}$$

Wavelet Estimator in the Haar basis:

$$H_{A,\Delta}(t) = \sum_{k=0}^{N/A-1} \left\{ A^{-1} \sum_{i=0}^{A-1} \Delta^{-1}(X_{t_{kA+i+1}} - X_{t_{kA+i}})^2 \right\} \mathbf{1}_{[\mathbf{t_{kA}},\mathbf{t_{(k+1)A}})}(\mathbf{t}) \tag{8}$$

Remarks :

* We have CMAE$_{A,\Delta}(t) = MAE_{A,\Delta}(t + A/2\Delta)$.
* The Estimator $MAE_{A,\Delta}(t)$ is defined for $t \geq A\Delta$, $CMAE_{A,\Delta}(t)$ for $t \in [A\Delta/2, T - A\Delta/2]$ and $H_{A,\Delta}(t)$ for every t.

* The Moving Average Estimator is the average of the A last intantaneous quadratic variation $\Delta^{-1}(X_{t_{j+1}} - X_{t_j})^2$, when the Haar basis Estimator is a package average.
* $\mathrm{MAE}_{A,\Delta}(t)$ is $(\mathcal{F}_t)$ adapted, CMAE(t) is $(\mathcal{F}_{t+A\Delta/2})$ adapted and for $t \in [kA\Delta,(k+1)A\Delta[$, $H_{A,\Delta}(t)$ is $(\mathcal{F}_{(k+1)A\Delta})$ adapted.
* By construction, the Estimators are piece-wise constant functions: $CMAE_{A,\Delta}(t)$ is constant on $I\!R - \Delta\mathbb{Z}$ and $H_{A,\Delta}(t)$ is constant on $I\!R - A\Delta\ \mathbb{Z}$.
* Since Moving Average Estimator is computed at each sampling time t_i and the Haar basis Estimator only for times t_{kA}, it is clear that the complexity of Numerical computation of Moving Average Estimator is A time more than for the Haar basis Estimator. But we make off-line estimation and this is not important.

Construction of the Moving Average Estimator:
The Kernel Estimator for (3) given in [11] is :

$$S_n(t) = n\left\{\sum_{i=1}^{n} K\left[(i/n)-t)h_n^{-1}\right]\right\}^{-1}\left\{\sum_{i=1}^{n} K\left[(i/n)-t)h_n^{-1}\right](X_{(i+1)/n} - X_{i/n})^2\right\}$$

where K is a compactly supported kernel. [11, Th.3, p.200] asserts that for a deterministic $\mathcal{C}^1$ function $\theta(.)$

$$(nh_n)^{1/2}\left[S_n(t) - \theta(t)^2\right] \Rightarrow \mathcal{N}(0, \|K\|^2_{L^2(SuppK)}\theta^2(t))$$

Let $\ell = mes(SuppK)$, we remark that $K_0 = \ell^{-1}1_{[0,\ell]}$ is an optimal kernel. Indeed, $S_n(.,\lambda K) = S_n(.,K)$, thus we can imposed the condition:

$$1 = \int_{SuppK} K(x)dx \leq \|K\|_{L^2(SuppK)}\|1\|_{L^2(SuppK)}$$

Therefore, $\|K\|_{L^2(SuppK)} \geq \ell^{-1/2} = \|K_0\|_{L^2(SuppK_0)}$. If we chose $\ell = 1, K = \mathbf{1}_{[-1,0]}$ and denote $\Delta = 1/n$ and $A = [nh_n]$, we obtain (6). Notice that Centred Moving Average Estimator corresponds to the symmetric kernel $K = \mathbf{1}_{[-1/2,1/2]}$.

Construction of the Wavelet Estimators:
Following [GLP], we consider $(V_j, j\in\mathbb{Z})$ a r-regular Multi Resolution Analysis of $L^2(I\!R)$ such that the associated scale function Φ and wavelet function ψ are compactly supported. For all j, the family $\left\{\Phi_{j,k}(t) = 2^{j/2}\,\Phi(2^jt-k), k\in\mathbb{Z}\right\}$ is an orthonormal basis of V_j. We have sampling time $\Delta = 2^{-n}$. For $j(n) < n$, the Wavelet Estimator is:

$$\hat{\theta}^2(t) = \sum_k \hat{\mu}_{j(n),k}\,\Phi_{j(n),k}(t) \tag{9}$$

where

$$\hat{\mu}_{j,k} = \sum_{i=0}^{N-1} \Phi_{j,k}(t_i)(X_{t_{i+1}} - X_{t_i})^2 \tag{10}$$

The r-regular Multi Resolution Analysis are convenient to approximate the functions $\theta(.) \in \mathcal{C}^m(I\!R)$, with m >r. The assumptions (A1) or (A2) induce us to consider a 0-regular analysis corresponding to the Haar basis with scale function $\Phi(t) = 1_{[0,1)}$. We denote $A = 2^{n-j(n)}$ and (9, 10) give us the Estimator (8).

3.2 Local Consistency of Estimators, Asymptotic Normality and Rate of Convergence.

Applying Itô formula on $(X_s - X_{t_i})^2$, we get the following decomposition which is an useful tool for all this paper.

Proposition 3.1 *Assume that (A0) is satisfied. Then*

$$S_{A,\Delta}(t) = M_{A,\Delta}(t) + N_{A,\Delta}(t) + D_{A,\Delta}(t) \tag{11}$$

where:

$$M_{A,\Delta}(t) = \sum_{j=0}^{N-1} \left\{ A^{-1} \sum_{i=-A/2}^{A/2} \overline{\theta}^2_{j+i} \right\} \mathbf{1}_{[t_j,t_{j+1})}(t) \tag{12}$$

$$N_{A,\Delta}(t) = 2\sum_{j=0}^{N-1} \left\{ A^{-1} \sum_{i=-A/2}^{A/2} \xi_{j+i} \right\} \mathbf{1}_{[t_j,t_{j+1})}(t) \tag{13}$$

$$D_{A,\Delta}(t) = 2\sum_{j=0}^{N-1} \left\{ A^{-1} \sum_{i=-A/2}^{A/2} \eta_{j+i} \right\} \mathbf{1}_{[t_j,t_{j+1})}(t) \tag{14}$$

where

$$\overline{\theta}^2_i = \Delta^{-1} \int_{t_i}^{t_{i+1}} \theta^2(s)ds. \tag{15}$$

$$\xi_i = \Delta^{-1} \int_{t_i}^{t_{i+1}} \theta(s) \left[\int_{t_i}^{s} \theta(u)\, dW_u \right] dW_s \tag{16}$$

$$\eta_i = \Delta^{-1} \left\{ \int_{t_i}^{t_{i+1}} b(s, X_s)(X_s - X_{t_i})\, ds + \int_{t_i}^{t_{i+1}} \sigma(s) \left[\int_{t_i}^{s} b(u, X_u)\, du \right] dW_s \right\} \tag{17}$$

Similarly

$$H_{A,\Delta}(t) = HM_{A,\Delta}(t) + HN_{A,\Delta}(t) + HD_{A,\Delta}(t) \tag{18}$$

and

$$HM_{A,\Delta}(t) = \sum_{k=0}^{N/A-1} \left\{ A^{-1} \sum_{i=kA}^{(k+1)A-1} \bar{\theta}_i^2 \right\} \mathbf{1}_{[t_{kA}, t_{(k+1)A})}(t) \tag{19}$$

$$HN_{A,\Delta}(t) = 2 \sum_{k=0}^{N/A-1} \left\{ A^{-1} \sum_{i=kA}^{(k+1)A-1} \xi_i \right\} \mathbf{1}_{[t_{kA}, t_{(k+1)A})}(t) \tag{20}$$

$$HD_{A,\Delta}(t) = 2 \sum_{k=0}^{N/A-1} \left\{ A^{-1} \sum_{i=kA}^{(k+1)A-1} \eta_i \right\} \mathbf{1}_{[t_{kA}, t_{(k+1)A})}(t) \tag{21}$$

Remarks :

* The functions $M(t)$ and $HM(t)$ only depend on $\overline{\theta}_i^2$, the average value of $\theta^2(t)$ on the sampling interval $[t_i, t_{i+1})$. The functions $N(t)$ and $HN(t)$ depend on the random variables ξ_i which properties are given in Appendix A. The functions $D(t)$ and $HD(t)$ contain all the terms coming from the drift and vanishes when $b(x,t) = 0$.

* When $b = 0$ and $\theta(t)$ is deterministic, $N_{A,\Delta}(t)$ corresponds to the variance term and $M_{A,\Delta}(t)$ to the bias term of Mean Integrated Square Error. In the general case, decomposition (11) differs from this classical decomposition. For instance, η_i and moreover $D_{A,\Delta}(t)$ contain variance terms (the stochastic integral) and bias terms (the Lebesgue Integral).

We study separately the three terms of the above decomposition.

Proposition 3.2 *We consider the asymptotic $A\Delta \to 0$*
(i) Assume that (A2) holds. Then for every t,

$$\mid M_{A,\Delta}(t) - \ \theta^2(t) \mid, \mid HM_{A,\Delta}(t) - \ \theta^2(t) \mid \leq K(\omega)(A\Delta)^m$$

(ii) Assume that (A1) holds. Then for every t , for $A\Delta$ small enough $M_{A,\Delta}(t) = \theta^2(t_-)$ (where $\theta^2(t_-)$ denote the left limit of $\theta^2(.)$) and at every t point of continuity of $\theta^2(.)$, $HM_{A,\Delta}(t) = \theta^2(t)$.

Proof: Obvious ■

Remark :

At a change point $HM_{A,\Delta}(\rho)$ could have no limit. But, $HM_{A,\Delta}(.)$ is the orthonormal projection from $\theta^2(.)$ on the space $V_j = \{f \text{ constant on } I\!R - 2^{-j}\mathbb{Z}\}$ and thus the better approximation of the function $\theta^2(.)$ by a piece-wise constant function in V_j.

The terms $D(.)$ and $HD(.)$ are of order $\Delta^{1/2}$, as stated below:

Proposition 3.3 *Assume that (A0) and (B1) are satisfied. Then for every $t > 0$*

$$I\!E \mid D_{A,\Delta}(t) \mid^2,\ I\!E \mid HD_{A,\Delta}(t) \mid^2 \leq 48\sqrt{2}\Delta \left[K_T \, \|E\theta^4(.)\|_{L^\infty(0,T)}\right]^{1/2} [1 + \mathcal{O}(\Delta)] \tag{22}$$

Proof: The two proofs are the same. From (13) and Jensen Inequality we get:

$$I\!E \mid D_{A,\Delta}(t) \mid^2 \leq 4A^{-1} \sum_{i=-A/2}^{A/2} I\!E(\eta_{j+i}^2)$$

The result follows from Lemma A.1 ■

The terms $N_{A,\Delta}(t)$ and $HN_{A,\Delta}(t)$ satisfy a Central Limit Theorem, as stated below.

Proposition 3.4 *(i) Assume that (A0) is satisfied. Then for every t*

$$I\!E \mid N_{A,\Delta}(t) \mid^2,\ I\!E \mid HN_{A,\Delta}(t) \mid^2 \leq 12A^{-1} \|E\theta^4(.)\|_{L^\infty(0,T)} \tag{23}$$

(ii) If, moreover $A \to +\infty$, $A\Delta \to 0$, and $\exists \nu_0 > 0$ such that $\forall t$, $\theta(t)^2 \geq \nu_0$. Then we have

$$A^{1/2}\, N_{A,\Delta}(t)/\theta^2(t_-) \Rightarrow \mathcal{N}(0, \sqrt{2}) \tag{24}$$

If, furthermore (A2) is satisfied or (A1) satisfied and $I\!P(\rho = t) = 0$. Then

$$A^{1/2}\, HN_{A,\Delta}(t)/\theta^2(t) \Rightarrow \mathcal{N}(0, \sqrt{2}) \tag{25}$$

Remarks :

* The condition $P(\rho = t) = 0$ means that almost surely t is not a volatility jump time.

* For $\theta(.)$ deterministic, the result follows from Lindeberg Theorem with Lyapunov Condition. Here the difficulty to obtain (ii) comes from $\theta(.)$ stochastic and we need to use Central Limit Theorem for martingale array.

* Lyapunov Condition is satisfied if and only if $A \to +\infty$, **this is another reason to emphasis dependence on Window Size.**

Proof: The two proofs are similar. We just prove Proposition for Moving Average Estimator.

(i) Using Lemma A.2 (43) we have $I\!\!E \mid N_{A,\Delta}(t) \mid^2 = 4A^{-2}\sum_{i=1}^{A} I\!\!E(\xi_j^2)$. Therefore (23) follows from (44).

(ii) **Proof when $\theta(.)$ is deterministic:**
In this case, using (46), we have:

$$4\sum_{i=1}^{A} A^{-2} I\!\!E(\xi_{j-i}^2) = 2A^{-2}\sum_{i=1}^{A}\left[\bar{\theta}_{j-i}^2\right]^2 = 2A^{-1}\mathcal{M}(A,\Delta,\bar{\theta}^2)(t_j)$$

by denoting $\mathcal{M}(A,\Delta,f)(.)$ the functional operator of Moving Average. From Lindeberg Theorem [Bi, th.27.3, p.371], we get:

$$A^{1/2}\left[\mathcal{M}_{A,\Delta}(\bar{\theta}^2)(t)\right]^{-1/2} N_{A,\Delta}(t) \Rightarrow \mathcal{N}(0,\sqrt{2})$$

But, when (A1) or (A2) holds, $\lim_{A\Delta->0}\mathcal{M}(A,\Delta,\bar{\theta}^2)(t) = \theta^2(t_-)$, we deduce (24). It remains to verify Lyapunov Condition. Using (47), we get:

$$A^2\mathcal{M}_{A,\Delta}(\bar{\theta}^2)(t_j)^{-2}\sum_{i=1}^{A} A^{-4} I\!\!E\xi_{j-i}^4$$

$$= A^{-1}\frac{15}{8}\mathcal{M}(A,\Delta,\bar{\theta}^2)(t_j)^{-2}A^{-1}\sum_{i=1}^{A}\left[\bar{\theta}_j^2\right]^2 \to 0 \qquad as\ A \to +\infty$$

(ii) **Proof for the stochastic case:**
We consider the time t_j as fixed and A,Δ depending on N as defined by the asymptotic. Let $S_N := \theta^{-2}(t_j - A\Delta)\, N_{A,\Delta}(t_j) = 2A^{-1}\sum_{i=1}^{A-1}\theta^{-2}(t_j - A\Delta)\,\xi_{j-i}$ and $\mathcal{F}_{N,i} := \mathcal{F}_{i\Delta/N}$, the random variables ξ_i are $\mathcal{F}_{N,i+1}$ adapted. Since $\bar{\theta}^2(t_j - A\Delta)$ is $\mathcal{F}_{N,j-i}$ adapted (for every $i \le A$) and $I\!\!E(\xi_i \mid \mathcal{F}_{N,i}) = 0$, S_N is a martingale array. Anyway the filtrations are not nested i.e. Condition (3.21) [18, p.58] is not satisfied. For discretized diffusion, this fact is point out in [16, p.121]. To avoid this difficulty, we prove that the Conditional Variance V_N^2 has, in Probability, a deterministic limit [18, p.59]. Indeed, we have:

$$V_N^2 := 4A^{-2}\sum_{i=j-A}^{j-1} I\!\!E(\theta^{-4}(t_j - A\Delta)\,\xi_i^2 \mid \mathcal{F}_{N,i}) =$$

$$= 4A^{-2}\,\theta^{-4}(t_j - A\Delta)\sum_{i=j-A}^{j-1} I\!\!E(\xi_i^2 \mid \mathcal{F}_{N,i})$$

From Lemma A.3 (49), we get:

$$I\!\!E(\xi_i^2 \mid \mathcal{F}_{N,i}) = \frac{1}{2}\theta^4(t_j - A\Delta) + \lambda_i$$

Therefore

$$V_n^2 = 2A^{-1}\left[1 + Rest\right]$$

with

$$Rest = 4A^{-1} \sum_{i=j-A}^{j-1} \theta^{-4}(t_j - A\Delta)\lambda_i$$

where exact formula for λ_i could be derived from Lemma A.3.

If (A2) holds, we have $\mathbb{E}(Rest^2) = \mathcal{O}(A\Delta)^{2m} \to 0$ as $A\Delta \to 0$. If (A1) holds, we have $\mathbb{E}(Rest^2) = \mathcal{O}(\mathbb{P}(\rho \in [t - A\Delta, t)^{1/4}) \to 0$ from (51). In both case, we apply Central Limit Theorem for array martingale [18, cor.3.1, p.58] (after normalization by $A^{1/2}$) and we get (24). The verification of Lyapunov Condition is the same as for deterministic case, replacing formula (46) by bound (44).

For CMAE or Haar basis Estimator, we have a bound with (for e.g. CMAE) $\mathbb{P}(\rho \in [t_{j_n} - A_n\Delta_n/2, t_{j_n} + A_n\Delta_n/2[) \to \mathbb{P}(\rho = t)$. For this reason, we need the extra condition $\mathbb{P}(\rho = t) = 0$ ■

The above results allow us to deduce the following theorem:

<u>THEOREM</u> 3.1 *Assume that (A0), (B1) are satisfied,* $A \to +\infty$ *and* $A\Delta \to 0$ *and* $\exists \nu_0 > 0$ *such that* $\forall t$, $\theta(t)^2 \geq \nu_0$. *Then*
1) $MAE_{A,\Delta}(t) \to \theta^2(t_-)$ *a.s. and, at every point of continuity of* $\theta^2(.)$, *we have a.s.* $H_{A,\Delta}(t) \to \theta^2(t)$ *and* $CMAE_{A,\Delta}(t) \to \theta^2(t)$.
2) If (A1) holds then

$$A^{1/2}\left[MAE_{A,\Delta}(t) - \theta^2(t_-)\right]/\theta^2(t_-) \Rightarrow \mathcal{N}(0, \sqrt{2})$$

If, moreover $\mathbb{P}(\rho = t) = 0$, *then for the two other Estimators (S=CMAE or H) , we have:*

$$A^{1/2}\left[S_{A,\Delta}(t) - \theta^2(t)\right]/\theta^2(t) \Rightarrow \mathcal{N}(0, \sqrt{2})$$

If (A2) holds and $\Delta A^{1+1/2m} \to 0$. *Then we have Asymptotic Normality of the three Estimators (S= MAE, CMAE or H) :*

$$A^{1/2}\left[S_{A,\Delta}(t) - \theta^2(t)\right]/\theta^2(t) \Rightarrow \mathcal{N}(0, \sqrt{2})$$

Proof :

We use decomposition (11) (resp. 18). From Proposition 3.2, when (A1) or (A2) is fulfilled each estimator converges to $\theta(t)^2$ or $\theta(t_-)^2$ a.s. From Proposition 3.3 and Proposition 3.4

the two remainding terms of the decomposition converge to 0 in L^2 norm. Moreover, using the bounds (23) and (22), we obtain almost sure convergence to 0. This conclude the proof of the first point of Theorem 3.1 (strong consistency of the estimators).

The second point follows from decomposition (11) (resp. 18) and the proposition 3.2, 3.3 and 3.4 ■

Remark :
The condition $\Delta A^{1+1/2m} \to 0$ is equivalent to $(A\Delta)^m \ll A^{-1/2}$ and is used to slight the error on the term $M(t)$ over the term $N(t)$. Elsewhere, we have convergence at rate $A\Delta$ without Asymptotic Normality.

For Centred Moving Average Estimator (or MAE), Theorem 3.1 generalizes the result of D.Florens-Zmirou [11, Th.3, p.200] from deterministic regular volatity $\theta(.)$ to the case of a stochastic (and less regular) volatility. The asymptotic $h_n \to 0$ and $nh_n \to +\infty$ becomes $A\Delta \to 0$ and $A \to +\infty$ in our notations and Condition $nh_n^3 \to 0$ [4, p.535] corresponds to $(A\Delta)^{2m} \ll A^{-1}$ for $m = 1$. For Wavelet Estimators, punctual convergence has not been studied before.

4 Comparison of Mean Integrated Square Error.

punctual convergence shows no difference between the three Estimators: they converge to the true value at the same rate. **In this section, we consider the sampling interval Δ as a fix small parameter and the size of the window A as a free parameter.** This corresponds to the situation in the financial applications. Haar basis Estimator is define by (8) only when N/A is an integer. But it can be easily generalized, since for $\nu := [N/A]$, the family $\left\{\mathbf{1}_{[\mathbf{k A\Delta},(\mathbf{k}+\mathbf{1})\mathbf{A\Delta}[},\ \mathbf{k} \in [\mathbf{0},\nu]\right\} \cup \left\{\mathbf{1}_{[\nu\mathbf{A\Delta},\mathbf{T}[}\right\}$ is orthogonal in $L^2(0,T)$. To compare them more precisely, we look at Integrated Square Error (ISE) and its mean value (MISE) :

<u>*Definition :*</u> *For a given weight function $\gamma(.) \geq 0$ and an Estimator $S(.)$, we define*

$$ISE(\gamma, A, \Delta) = \int_0^T \gamma(t)\left[S(t) - \theta^2(t)\right]^2 dt$$

$$MISE(\gamma, A, \Delta) = \mathbb{E}\int_0^T \gamma(t)\left[S(t) - \theta^2(t)\right]^2 dt$$

Let us precise some notations. Without any loss of generality we assume that $\gamma(.)$ is piece-wise constant on the intervals $[t_j, t_{j+1})$ with value 0 or 1 (every continuous function is bounded on a compact set by such a function multiplied by its L^∞ norm). We assume that Supp γ is included in the definition set of the Estimators (CMAE, MAE). We define:

$$R1(\gamma, A, \Delta) = \int_0^T \gamma(t) \left[M_{A,\Delta}(t) - \theta^2(t)\right]^2 dt$$

$$R2(\gamma, A, \Delta) = \int_0^T \gamma(t) N^2_{A,\Delta}(t)\, dt$$

We add an index H, MA or MAE to denote MISE, R1, R2 ... for Haar basis Estimator, MAE or CMA. Remark that, in the deterministic case without drift, $\mathbb{E}(R_1)$ corresponds to the bias term and $\mathbb{E}(R_2)$ to the variance term, see e.g. [19, p.233]. This does not hold in the general case.

4.1 A typical case.

We first give MISE in a simple but meanigful case in which we have an explicit formula for $R1(\gamma, A, \Delta)$ and we can easily compare the three Estimators: there is at most one volatility jump time in each window, $\theta(.)$ is deterministic and there is no drift ($b = 0$).

Proposition 4.1 *Assume that (A3) is satisfied, $\theta(.)$ is deterministic and $b = 0$. Then for $\gamma = \mathbf{1}_{[\mathbf{A\Delta}, \mathbf{T-A\Delta}[}$ and if no volatility jump time belongs to $[0, A\Delta[\cup[T - 2A\Delta, T]$*
1) For Moving Average Estimators (MAE and CMAE), we have

$$MISE^{CMA}(A, \Delta) = \rho_{CMA}\, A\Delta\, (1 + \frac{3}{2}A^{-1} + \frac{1}{2}A^{-2}) + 2A^{-1}L_{CMA}(A)$$

$$MISE^{MA}(A, \Delta) = 4\rho_{CMA}\, A\Delta\, (1 + \frac{3}{2}A^{-1} + \frac{1}{2}A^{-2}) + 2A^{-1}L_{MA}(A)$$

with

$$L_{MA}(A) = \sum_{j=0}^{N-1} \alpha_j \Delta\, \theta^4(t_j) \qquad and \qquad L_{CMA}(A) := \sum_{j=0}^{N-1} \beta_j \Delta\, \theta^4(t_j)$$

where α_j and β_j are given in Proposition 4.2 and $\rho_{CMA} := \frac{1}{12}\sum_{\rho=0}^{f} \gamma(\rho) \left[\delta\theta^2_\rho\right]^2$
2) For Haar basis Estimator, we have

$$MISE^H(A, \Delta) = \rho_H(A) A\Delta + 2A^{-1}L_H$$

with

$$L_H := \sum_{j=0}^{N-1} \Delta\gamma(t_j)\, \theta^4(t_j); \qquad \rho_H(A) := \sum_{j=0}^{N-1} \gamma(t_\rho) x_\rho (1 - x_\rho) \left[\delta\theta^2_\rho\right]^2$$

and $x_\rho := (\rho - kA)/A \in [0, 1]$ for $t_\rho \in [t_{kA}, t_{(k+1)A}[$.

Proof:

This results from Proposition 4.2 below and (A3) wich induces, for each volatility jump time t_ρ , θ_j^2 is constant on $[t_\rho,\ t_\rho + A\Delta]$. Hence, for MAE for example, we have:

$$\sum_{\rho=0}^{f}\sum_{j=\rho}^{\rho+A}\gamma(t_j)\Delta\left[\theta_\rho^2 - A\sum_{i=1}^{A-1}\theta_{j-i}^2\right]^2 = \Delta A^{-2}\sum_{\rho=0}^{f}\gamma(t_\rho)\sum_{k=1}^{A}\left[k\delta\theta_\rho^2\right]^2$$

$$= (A\Delta)\frac{(A+1)(2A+1)}{6A^2}\sum_{j=0}^{N-1}\gamma(t_\rho)\left[\delta\theta_\rho^2\right]^2$$

Comparison of Haar basis Estimator and Centred Moving Average Estimator (CMAE):

We have $L_H = \int_0^T \theta^4(s)\gamma(s)\,ds$ and $L_{CMA}(A) = L_H + \mathcal{O}(\Delta)$, therefore CMAE is better than Haar basis Estimator as soon as $\rho_H > \rho_{CMA}$. This closely depends on the positions of the change times near the point t_{kA}: we have $x(1-x) > 1/12$ iff $x \in (0.09; 0.91)$. For 82% of cases we get $\rho_H > \ \rho_{CMA}$. If we replace $x_j(1-x_j)$ by its mean value $1/6$, we get $\bar\rho_H = 2\rho_{CMA} = \ \frac{1}{2}\ \rho_{MA}$. This explain why CMAE is better than Haar basis Estimator in most circumstances. Moreover the function $\rho_H(A)$ is oscillating, this explains the same shape of MISE$^H(A,\Delta)$, see picture 6.4. From the other hand, we have $x_j(1-x_j) \leq 1/4 < 1/3$, thus $\rho_H < 4\,\rho_{CMA}$ in every case and $MISE^H(A,\Delta)$ is always smaller for Haar basis Estimator than $MISE^{MA}(A,\Delta)$ for MAE.

* We have a delay of $\Delta A/2$ for CMAE and a delay between 0 and $A\Delta$ for Haar basis Estimator, thus the same average delay.

* In general case, $R1^H(A,\Delta)$ is hardly oscillating function on A as shown by numerical simulations (Pictures 6.4, 6.5 and 6.6).

Comparison of MAE and CMAE:

The bias term $\mathbb{E}R1^H(A,\Delta)$ is four time greater for MAE than for CMAE whereas the viariances are almost the same. This explain how CMAE is better than MAE.

Moreover, when (A3) holds, it is easy to prove that CMAE is optimal for MISE in the class of Estimators obtained as delayed MAE. This justify its use. Recall that Estimator with constant Kernel are in a certain sense optimal Kernel Estimators (see section 2.1).

The assumption (A3) holds only for A lesser than a critical value A_c. For $A < A_c$ the bias term linearly depends on A. However, using (26) below, we can prove that $\mathbb{E}R1(A,\Delta)$ is bounded by $(1+\parallel \gamma \parallel_\infty) \parallel \theta^2 \parallel^2_{L^2(O,T)}$.

An interesting approximation:

For $A < A_c$, θ^2 is constant on $[t_{j-A}, t_j]$. In this case, we can compute MISE(A, Δ) and its "*derivative*" with respect to A for both MAE and CMAE. We find $MISE^{MA}(A, \Delta)$ is a convex function on A with a unique minimum :

$$A_{opt}^{MA}(\Delta) = \Delta^{-1/2}(2L_H/\rho_{ma})^{1/2}\,[1 + \mathcal{O}(\Delta)]$$

$$MISE_{opt}^{MA}(\Delta) = \Delta^{1/2}(8\rho_{MA}\,L_{MA})^{1/2}\,[1 + \mathcal{O}(\Delta)]$$

and

$$MISE_{opt}^{CMA}(\Delta) = \frac{1}{2}MISE_{opt}^{MA}(\Delta) + \mathcal{O}(\Delta^{3/2})$$

$$A_{opt}^{CMA}(\Delta) = 2A_{opt}^{MA}(\Delta) + \mathcal{O}(\Delta^{1/2})$$

It is interesting to denote that optimal value of $MISE^{CMA}(A, \Delta)$ depends on $\mathbf{\Delta^{1/2}}$. Indeed $MISE^{CMA}(A, \Delta) \geq Cte\,\Delta^{1/2}$, when (A3) holds it is possible to prove it. We wonder it is true for every function $\theta(.)$. Remark that if there is no volatility change (i.e. θ is constant) then $\rho_{MA} = 0$. Therefore $A_{opt} = +\infty$ and $MISE_{opt} = 0$. There is a great qualitative change between the time constant case (parametric case) and the time varying case (non-parametric case).

4.2 General case.

In the stochastic case, we have the following formulas for R_1 and $\mathbb{E}R_2$:

Proposition 4.2 *: Assume that (A0) is satisfied. Then*
1) For Moving Average Estimator

$$R1^{MA}(\gamma, A, \Delta) = \Delta\sum_{j=0}^{N-1}\gamma(t_j)V_j^2 + \Delta\sum_{j=0}^{N-1}\gamma(t_j)\left[\bar{\theta}_j^2 - A^{-1}\sum_{i=1}^{A-1}\bar{\theta}_{j-i}^2\right]^2 \tag{26}$$

and

$$\mathbb{E}R2^{MA}(\gamma, A, \Delta) = 4A^{-1}\sum_{j=0}^{N-1}\alpha_j\Delta\mathbb{E}(\xi_j^2) \tag{27}$$

where

$$V_j^2 := \Delta^{-1}\int_{t_j}^{t_{j+1}}\left[\theta^2(s) - \bar{\theta}_j^2\right]^2 ds \qquad \text{and} \qquad \alpha_j := A^{-1}\sum_{i=1}^{A}\gamma(t_{j+i}) \tag{28}$$

2) For Centred Moving Average Estimator

$$R1^{CMA}(\gamma, A, \Delta) = \Delta\sum_{j=0}^{N-1}\gamma(t_j)V_j^2 + \Delta\sum_{j=0}^{N-1}\gamma(t_j)\left[\bar{\theta}_j^2 - A^{-1}\sum_{i=j-A/2}^{j+A/2}\bar{\theta}_i^2\right]^2 \tag{29}$$

$$\mathbb{E}R2^{CMA}(\gamma, A, \Delta) = 4A^{-1} \sum_{j=0}^{N-1} \beta_j \Delta\, \mathbb{E}(\xi_j^2) \tag{30}$$

with $\beta_j = A^{-1} \sum_{i=j-A/2}^{j+A/2} \gamma(t_i)$

3) For Haar basis Estimator:

$$R1^H(\gamma, A, \Delta) = \Delta \sum_{j=0}^{N-1} \gamma(t_j) V_j^2 + \Delta \sum_{k=0}^{N/A-1} \sum_{j=kA}^{(k+1)A-1} \gamma(t_j) \left[\bar{\theta}_j^2 - A^{-1} \sum_{i=kA}^{(k+1)A-1} \bar{\theta}_i^2\right]^2 \tag{31}$$

$$\mathbb{E}R2^H(\gamma, A, \Delta) = 4A^{-1} \sum_{j=0}^{N-1} \varphi_j \Delta \mathbb{E}(\xi_j^2) \tag{32}$$

with $\varphi_j = \sum_{i=kA}^{(k+1)A-1} \gamma(t_i)$ *where* $kA \leq j < (k+1)A$

Proof: The formulas (27, 30 and 32) follows from (2.14) and (A2.1).

To prove (26, 29 and 31), consider for example $R1^{MA}(\gamma, A, \Delta)$ and let $\mu_j = A^{-1} \sum_{i=1}^{A} \bar{\theta}_{j-i}^2$, we get :

$$R1^{MA}(\gamma, A, \Delta) = \sum_{j=0}^{N-1} \gamma(t_j) \int_{t_j}^{t_{j+1}} \left[\mu_j - \theta^2(s)\right]^2 ds$$

and

$$\int_{t_j}^{t_{j+1}} \left[\mu_j - \theta^2(s)\right]^2 ds = \Delta V_j^2 + \Delta \left[\mu_j - \bar{\theta}_j^2\right]^2$$

■

Remarks :

* If (A1) holds we have $V_j^2 \leq \frac{1}{4}\left[\delta\theta_j^2\right]^2$. If (A2) holds $V_j^2 \leq K^2(\omega)\Delta^{2m}$. Therefore, replacing $\theta^2(t)$ by its average values $\bar{\theta}_j^2$ induces on $R1(\gamma, A, \Delta)$ an error of $\Delta \sum_{j=0}^{N-1} \gamma(t_j) V_j^2$, which is bounded either by $\frac{1}{4}\Delta \sum_{\rho=0}^{f} \left[\delta\Delta\theta_\rho^2\right]^2$ either by $T\,K^2(\omega)\Delta^{2m}$ and in both case is of smaller order than the remaining terms.

* All the above formulas are written in a convenient way to do numerical simulations.

To notational convenience, let $\gamma = \mathbf{1}_{[\mathbf{A\Delta}, \mathbf{T-A\Delta}]}$. We get:

$$\begin{aligned} \mid \mathbb{E}R2^H(A) - \mathbb{E}R2^{CMA}(A) \mid &= 4A^{-1}\Delta \sum_{j=1}^{A} (\frac{j}{A}) \left[\mathbb{E}(\xi_j^2) + E(\xi_{N-j}^2)\right] \\ &\leq 12\Delta\,(1 + A^{-1}) \parallel E\theta^4 \parallel_\infty \end{aligned}$$

In every case, we have $\mathbb{E}R2^{CMA}(A) = \mathbb{E}R2^H(A) + \mathcal{O}(\Delta)$ and $\mathbb{E}R2^{MA}(A) = \mathbb{E}R2^H(A) + \mathcal{O}(\Delta)$ for every A. Therefore, the difference $\mid \mathbb{E}R2^H(A) - \mathbb{E}R2^{CMA}(A) \mid$ could be disregarded (even without knowing $\mathbb{E}(\xi_j^2)$ as in the deterministic case). The main difference between Centred Moving Average Estimator and Haar basis Estimator comes from the difference $\|\theta^2 - \mathcal{M}^H(A,\Delta,\bar{\theta}^2)\|^2_{L^2(O,T;\gamma dt)} - \|\theta^2 - \mathcal{M}^{CMA}(A,\Delta,\bar{\theta}^2)\|^2_{L^2(O,T;\gamma dt)}$, where $\mathcal{M}^{CMA}(A,\Delta,.)$ (respectively $\mathcal{M}^H(A,\Delta,.)$) denotes the linear operator of Centred moving average (resp. packed average). This difference only depends on $\bar{\theta}_j^2$, the mean value of volatility on each sampling interval. This fact allows us to assume (A1'), without any loss of generality. Anyway, we cannot see the local fluctuation of volatility between to sampling times t_i and t_{i+1}. This comes from the discretized observation of the path.

We essentially have two different cases: isolated jumps of volatility (as in section 4.1) or time varying volatility. When at least one volatility jump occurs, the numerical simulations (pictures 6.5 and 6.6) show the same shape as for the isolated volatility jump case, i.e. $R1^H(A,\Delta)$ is an oscillating function on A and is most often larger than $R1^{CMA}(A,\Delta)$.

When (A1') holds, we have in both case

$$R1(A,\Delta) = \|\bar{\theta}^2 - \mathcal{M}(A,\Delta,\bar{\theta}^2)\|^2_{L^2(O,T;\gamma dt)}$$

Since $\|\mathcal{M}(A,\Delta,\bar{\theta}^2)\|^2_{L^2(O,T;\gamma dt)} \le \|\gamma\|_\infty \|\theta^2\|^2_{L^2(O,T)}$, we deduce the boundness of $R1(A,\Delta)$. For a general function $\theta^2(.)$ we have no precise study of the functions

$\mathbb{E}R1(A,\Delta) + \mathbb{E}R2(A,\Delta)$ as in section 5.1. We just have made some numerical simulations (pictures 5.5 and 5.6). However, we conjecture that Δ could disregarded behind $\inf_{A\in\mathbb{N}^*}[\mathbb{E}\ R1(A,\Delta) + \mathbb{E}\ R2(A,\Delta)]$.

If the last conjecture and (B1) hold, then the following proposition allows us to disregarded it for MISE.

Proposition 4.3 *Assume that (A0) and (B1) are satisfied. Then:*
1) For the Centred Moving Average Estimators (or MAE) and γ *constant on* $[t_i,\ t_{i+1})$

$$MISE(\gamma,A,\Delta) = \mathbb{E}R1(\gamma,A,\Delta) + \mathbb{E}R2(\gamma,A,\Delta) + \epsilon \tag{33}$$

where

$$|\ \epsilon\ | \le Cte\,\Delta\ [\mathbb{E}R1(\gamma,A,\Delta) + \mathbb{E}R2(\gamma,A,\Delta)]^{1/2} \tag{34}$$

2) For the Haar basis Estimator, if moreover, γ *is constant on the interval* $[kA\Delta,(k+1)A\Delta[$ *or* $\theta(.)$ *deterministic, then (33) holds.*

Proof:

We just prove the Haar basis case, the other being more simple. It results from the two following points:

From Proposition 3.4 :

$$\mathbb{E}\int_O^T \gamma(t) HD^2_{A,\Delta}(t)\,dt \le \Delta\, Cte\, \|\gamma\|_{L^1(0,T)}$$

From the other hand

$$\mathbb{E}\int_O^T \gamma(t)\, HN_{A,\Delta}(t)\left[\theta^2(t) - HM_{A,\Delta}(t)\right] dt = 0$$

Indeed, in the first case

$$\int_O^T \gamma(t)\, HN_{A,\Delta}(t)\left[\theta^2(t) - HM_{A,\Delta}(t)\right] dt$$

$$= \sum_{k=0}^{N/A-1}\left\{A^{-1}\sum_{i=kA}^{(k+1)A-1}\xi_i\right\}\int_{kA\Delta}^{(k+1)A\Delta}\gamma(s)\left[\theta^2(s) - A^{-1}\sum_{i=kA}^{(k+1)A-1}\bar{\theta}^2\right] ds$$

and by (15)

$$\int_{kA\Delta}^{(k+1)A\Delta}\gamma(s)\left[\theta^2(s) - A^{-1}\sum_{i=kA}^{(k+1)A-1}\bar{\theta}^2\right] ds = 0$$

When $\theta(.)$ is deterministic, we use $\mathbb{E}\xi_i = 0$, $\forall i$ to deduce $\mathbb{E} HN_{A,\Delta}(t) = 0$ ■

5 Cental Limit Theorem for Integrated Square Error in the Deterministic Case.

To make numerical simulations of the differents Estimators, we make simulation of a sample path of the S.D.E. with a known input function $\theta(.)$. In this case, $\theta(.)$ appears as a deterministic function, we have simpler stochastic calculations and we obtain Central Limit Theorem for ISE.

<u>THEOREM</u> 5.1 (CLT for ISE, Haar basis Estimator)
Assume that (A0) is satisfied and $\theta(.)$ is deterministic. Then when $\Delta \to 0$

$$A\,\Delta^{-1/2}\lambda_H^{-1}\left[R2^H(\gamma, A, \Delta) - 2A^{-1}\sum_{j=0}^{N-1}\Delta\,\gamma(t_j)(\bar{\theta}_j^2)^2\right] \Rightarrow \mathcal{N}(0,1) \tag{35}$$

where $\lambda_H^2 = 56\sum_{j=0}^{N-1}\Delta\gamma(t_j)(\bar{\theta}_j^2)^4$

Remarks:

* Here the asymptotic is $\Delta \to 0$. This is more less restrictive than in Theorem 3.1, where we need $A \to +\infty$ and $A\Delta \to 0$.
* Unformaly, we could rewrite (35) as

$$R2^H(\gamma, A, \Delta) = 2A^{-1} \sum_{j=0}^{N-1} \Delta\gamma(t_j)(\bar{\theta}_j^2)^2 + \lambda_H \Delta^{1/2} A^{-1} \mathcal{N}(0,1)$$

and $\lambda_H^2 \approx 56 \int_0^T \theta^8(s)\,\gamma(s)\,ds$. So the error between the functions $MISE^H(A,\Delta)$ and $ISE^H(A,\Delta)$ is $\lambda_H \Delta^{1/2} A^{-1} \mathcal{N}(0,1)$

Proof : We have

$$R2^H(\gamma, A, \Delta) = 4\,(A\Delta) \sum_{k=0}^{N/A-1} \gamma(t_{kA})\zeta_k^2$$

where $\zeta_k := A^{-1} \sum_{j=kA}^{(k+1)A-1} \xi_j$. The (ζ_k) and (ξ_j) are families of independent random variables.

We denote

$$U_n^2 = 16(A\Delta)^2 \sum_{k=0}^{N/A-1} \mathbb{E}\left[\zeta_k^2 - \mathbb{E}\zeta_k^2\right]^2 = 16(A\Delta)^2 \sum_{k=0}^{N/A-1} \mathbb{E}\zeta_k^4 - \left[\mathbb{E}\zeta_k^2\right]^2$$

From Lemma A.2, we get:

$$\mathbb{E}\zeta_k^2 = A^{-2} \sum_{j=kA}^{(k+1)A-1} \mathbb{E}\xi_j^2 = \frac{1}{2}A^{-2} \sum_{j=kA}^{(k+1)A-1} \bar{\theta}_j^2$$

and

$$\mathbb{E}R2^H(\gamma, A, \Delta) = 2A^{-1} \sum_{j=0}^{N-1} \Delta\,\bar{\theta}_j^2$$

Using the independency of (ξ_j) and $\mathbb{E}\xi_j = 0$ (see Lemma A.2), we get:

$$\mathbb{E}\zeta_k^4 = A^{-4} \sum_{j_1,j_2,j_3,j_4} \mathbb{E}(\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4}) = A^{-4} \sum_{j_1,j_2} \mathbb{E}(\xi_{j_1}^2 \xi_{j_2}^2)$$

Therefore

$$\begin{aligned}\mathbb{E}\zeta_k^4 - \left[\mathbb{E}\zeta_k^2\right]^2 &= A^{-4} \sum_{j_1,j_2} \left\{\mathbb{E}(\xi_{j_1}^2 \xi_{j_2}^2) - \mathbb{E}(\xi_{j_1}^2)\mathbb{E}(\xi_{j_2}^2)\right\} \\ &= A^{-4} \sum_{j=kA}^{(k+1)A-1} \left\{\mathbb{E}(\xi_j^4) - \mathbb{E}(\xi_j^2)^2\right\}\end{aligned}$$

and

$$U_n^2 = A^{-2}\Delta \sum_{j=0}^{N-1} \Delta \left[\mathbb{E}(\xi_j^4) - \mathbb{E}(\xi_j^2)^2 \right]$$

Using (46, 47), we get :

$$U_n^2 = 56\, A^{-2}\Delta \sum_{j=0}^{N-1} \Delta(\bar{\theta}_j^2)^4 = A^{-2}\Delta\, \lambda_H^2$$

Formula (35) results from Lindeberg Theorem after having verified Lyapunov Condition, which is equivalent to:

$$\lim_{\Delta \to 0} \Delta^2 \sum_{k=0}^{N/A-1} A^8\, \mathbb{E}\left[\zeta_k^2 - \mathbb{E}\zeta_k^2\right]^4 = 0$$

this follows from

$$\Delta^2 \sum_{k=0}^{N/A-1} A^8\, \mathbb{E}\left[\zeta_k^2 - \mathbb{E}\zeta_k^2\right]^4 = \Delta\, Cte \sum_{j=0}^{N-1} \Delta(\bar{\theta}_j^2)^4$$

(see Appendix B) ■

To prove Central Limit Theorem for Centred Moving Average Estimator (or MAE), we used the following assumptions, which is fulfilled when $\theta^2(.)$ is strictly positive:

(A4) $\exists \nu_0 > 0$ *such that* $\forall j \in [0, N]$, $\bar{\theta}_j^2 \geq \nu_0$.

<u>THEOREM</u> 5.2 (CLT for ISE, CMAE)

Assume that (A0,A4) are satisfied and $\theta(.)$ is deterministic. Then when $\Delta \to 0$

$$A^{1/2}\Delta^{-1/2}\lambda_{MA}^{-1} \left[R2^{CMA}(\gamma, A, \Delta) - 2A^{-1} \sum_{j=0}^{N-1} \Delta\gamma(t_j)(\bar{\theta}_j^2)^2 \right] \Rightarrow \mathcal{N}(0,1) \tag{36}$$

where

$$\lambda_{MA}^2 = \sum_{i=0}^{N-1} \Delta A^{-3} \left\{ 56\, (\bar{\theta}_i^2)^4\, \beta^2(i,i) + 16\, (\bar{\theta}_i^2)^2 \sum_{j<i} \beta^2(i,j)\, (\bar{\theta}_j^2)^2 \right\}$$

and $\beta(i,j)$ is the symmetric band array given by (37).

Remarks:

* Once again, the asymptotic is $\Delta \to 0$
* λ_{MA} depends on A and Δ, but it is bounded by $\|\theta^8\|_{L^\infty(0,T)}\, T\, (16 + 56A^{-1})$.

* We could rewrite (36) as

$$R2^{CMA}(\gamma, A, \Delta) = 2A^{-1} \sum_{j=0}^{N-1} \Delta(\bar{\theta}_j^2)^2 + \lambda_{MA} A^{-1/2} \Delta^{1/2} \mathcal{N}(0,1)$$

So the error between the functions $MISE^{CMA}(A)$ and $ISE^{CMA}(A)$ is

$\lambda_{MA}\, A^{-1/2}\Delta^{1/2}\,\mathcal{N}(0,1)$ and is greater than for the Haar basis Estimator.

* If $\theta^2(.)$ is a $\mathcal{C}^2$ function, Theorem 5.2 combined to Proposition 3.2 give us back the result of V.Lacoste [24, Th., p.320], after having changed the notations.

Proof: To simplify the notation, we work on MAE. Recall that CMAE is just MAE with a delay. We have:

$$R2^{MA}(\gamma, A, \Delta) = 4\Delta \sum_{j=0}^{N-1} \gamma(t_j) \left\{ A^{-1} \sum_{i=1}^{A-1} \xi_{j-i} \right\}^2 = 4\Delta A^{-2} \sum_{i,j=0}^{N-1} \beta(i,j)\, \xi_i \xi_j$$

where

$$\beta(i,j) := \sum_{k-A \leq i,j < k} \gamma(t_k) \tag{37}$$

We obviously have:

$$0 \leq \beta(i,j) \leq A \qquad and \qquad \beta(i,j) = 0 \; if \mid i-j \mid > 2A \tag{38}$$

For notational simplicity, we assume that (A1') holds. This induces no loss of generality (see lemma A.2). From (13) and (46), we get:

$$\mathbb{E} R2^{MA}(\gamma, A, \Delta) = 4\Delta A^{-2} \sum_{i=0}^{N-1} \beta(i,i) \mathbb{E}\xi_i^2 = 2A^{-1} \sum_{i=0}^{N-1} \alpha_i \Delta(\bar{\theta}_j^2)^2$$

Denote

$$S_N := R2^{MA}(\gamma, A, \Delta) - \mathbb{E} R2^{MA}(\gamma, A, \Delta) = 4\Delta\, A^{-2} \sum_{i=0}^{N-1} X_{N,i}$$

where

$$X_{N,i} := \beta(i,i) \left[\xi_i^2 - \mathbb{E}\xi_i^2\right] + 2\xi_i \sum_{j<i} \beta(i,j) \xi_j$$

Let $\mathcal{F}_i := \mathcal{F}_{iT/N}$ and $v_i := \sum_{j<i} \beta(i,j)\xi_j$, the random variables ξ_i are $\mathcal{F}_{i+1}$ adapted, so v_i is $\mathcal{F}_i$ adapted and ξ_i is independent from $\mathcal{F}_i$. Thus $\mathbb{E}(X_{N,i} \mid \mathcal{F}_i) = 0$ and S_N is a martingale array. Using Lemma A.2 47, we get :

$$\begin{aligned}
\mathbb{E}(X_{N,i}^2 \mid \mathcal{F}_i) &= \beta^2(i,i) \left[\mathbb{E}\xi_i^4 - (\mathbb{E}\xi_i^2)^2\right] + 4v_i \beta(i,i) \mathbb{E}\left[\xi_i(\xi_i^2 - \mathbb{E}\xi_i^2)\right] + 4\mathbb{E}(\xi_i^2)\, v_i^2 \\
&= \frac{7}{2}\, \theta^8(t_i)\, \beta^2(i,i) + 4v_i\, \beta(i,i)\, \theta^6(t_i) + 2v_i^2\, \theta^4(t_i)
\end{aligned}$$

Therefore

$$\begin{aligned} V_N^2 &:= 16\Delta^2 A^{-4} \sum_{i=0}^{N-1} \mathbb{E}(X_{N,i}^2 \mid \mathcal{F}_i) \\ &= \Delta^2 A^{-4} \sum_{i=0}^{N-1} \left\{ 56\,\beta^2(i,i)\,\theta^8(t_i) + 64\,\beta(i,i)\,\upsilon_i\,\theta^6(t_i) + 32\,\upsilon_i^2\,\theta^4(t_i) \right\} \end{aligned}$$

and

$$\mathbb{E}V_N^2 = \Delta^2 A^{-4} \sum_{i=0}^{N-1} \left\{ 56\,\beta^2(i,i)\,\theta^8(t_i) + 16\theta^4(t_i) \sum_{j<i} \beta^2(i,j)\,\theta^4(t_j) \right\}$$

From (38) and (A4), we deduce

$$k_0 \nu_0 \Delta A^{-1} \le \mathbb{E}V_N^2 \le k_1\,\Delta\,A^{-1}\,\|\theta^8\|_\infty$$

We define the normalized martingale array $\tilde{S}_N$ corresponding to $\tilde{X}_{N,i} := (\mathbb{E}V_N^2)^{-1/2} X_{N,i}$. We will show that $\tilde{V}_{N,i} := (EV_N^2)^{-1}\,V_N^2$ converges to 1 in Probability. From Central Limit Theorem for Martingale array [18, Cor.3.1, p.58], we get (36). To obtain convergence in Probability, it suffices to prove convergence in L^2 norm. We have:

$$\begin{aligned} \mathbb{E}\left[(\mathbb{E}V_N^2)^{-1} V_N^2 - 1 \right]^2 &\le Cte\,A^2\,\Delta^{-2} \mathbb{E}\left[V_N^2 - \mathbb{E}V_N^2 \right]^2 \\ &\le Cte\,\Delta^2 A^{-6} \mathbb{E}\left\{ \sum_{i=0}^{N-1} 2\beta(i,i)\upsilon_i\,\theta^6(t_i) + \theta^4(t_i)(\upsilon_i^2 - \mathbb{E}\upsilon_i^2) \right\}^2 \end{aligned}$$

Let $\zeta_i := 2\beta(i,i)\upsilon_i\,\theta^6(t_i) + \theta^4(t_i)\,(\upsilon_i^2 - \mathbb{E}\upsilon_i^2)$. From (38), we have $\zeta_i\,\zeta_j = 0$ as soon as $\mid i-j \mid > 2\,A$. Therefore

$$\begin{aligned} \mathbb{E}\left[(\mathbb{E}V_N^2)^{-1} V_N^2 - 1 \right]^2 &\le Cte\,\Delta^2\,A^{-6} \sum_{i=0}^{N-1} \sum_{j=i-A}^{i-1} \mathbb{E}(\zeta_i^2)^{1/2} \mathbb{E}(\zeta_j^2)^{1/2} \\ &\le \Delta\,Cte \to 0 \qquad as\ \Delta \to 0 \end{aligned}$$

Indeed, from (38) and Lemma A.2, we get:

$$\begin{aligned} \mathbb{E}(\zeta_i^2) &\le 4\,\beta^2(i,i)\,\theta^{12}(t_i)\,\mathbb{E}(\upsilon_i^2) + \theta^8(t_i)\,\mathbb{E}\upsilon_i^4 + 4\,\beta(i,i)\,\theta^{12}(t_i)\,\mathbb{E}(\upsilon_i^3) \\ &\le Cte\,\|\theta^{16}\|_{L^\infty(0,T)} \left\{ \sum_{j=i-A}^{i-1} \beta^4(i,i) + \sum_{j=i-A}^{i-1} \beta^3(i,i) \right\} \\ &\le Cte\,A^5\,\|\theta^{16}\|_{L^\infty(0,T)} \end{aligned}$$

It remains to verify Lyapunov Condition, this is done in Appendix B ■

6 Numerical Study.

We take $T = 1$ and fix $N = 5000$, $\Delta = 2.10^{-3}$ and $\gamma = \mathbf{1}_{[\mathbf{A\Delta},\mathbf{T-A\Delta}]}$. We consider different cases of deterministic function $\theta(.)$ satisfying (A1'). From Proposition 3.2, we have :

$$MISE^{MA}(A,\Delta) = \Delta \sum_{j=A}^{N-A} \left[\theta^2(t_j) - A^{-1}\sum_{i=1}^{A-1}\theta^2(t_{j-i})\right]^2 + 2A^{-1}\sum_{j=0}^{N-1}\alpha_j\Delta\theta^4(t_j) \tag{39}$$

$$MISE^{CMA}(A,\Delta) = \Delta \sum_{j=A}^{N-A} \left[\theta^2(t_j) - A^{-1}\sum_{i=j-A/2}^{j+A/2}\theta^2(t_i)\right]^2 + 2A^{-1}\sum_{j=0}^{N-1}\beta_j\Delta\theta^4(t_j) \tag{40}$$

$$MISE^{H}(A,\Delta) = \Delta \sum_{k=1}^{N/A-2}\sum_{j=kA}^{(k+1)A-1} \left[\theta^2(t_j) - A^{-1}\sum_{i=kA}^{(k+1)A-1}\theta^2(t_i)\right]^2 + 2A^{-1}\sum_{j=A}^{N-A}\Delta\theta^4(t_j) \tag{41}$$

We consider three examples of function $\theta(.)$:

Example 6.1: Few volatility jumps (two jumps)

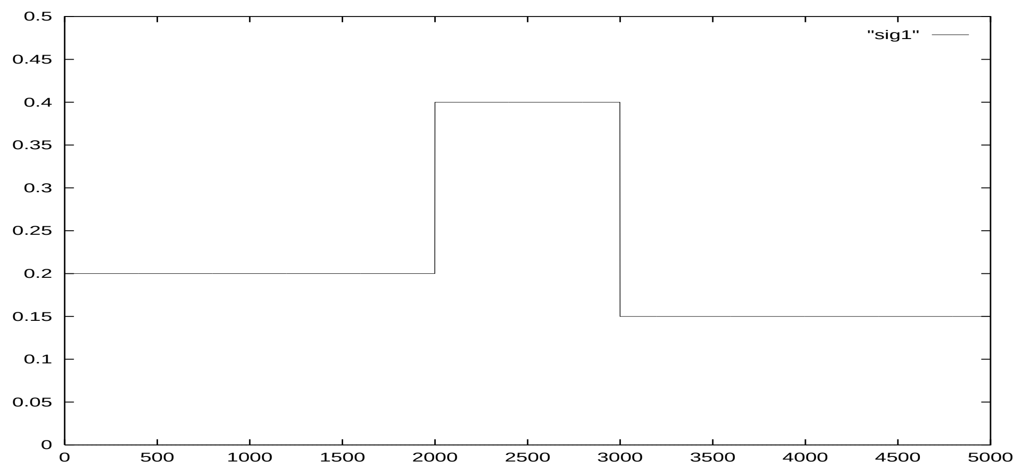

fig 6.1

Example 6.2: A lot of volatility jumps (19 jumps)

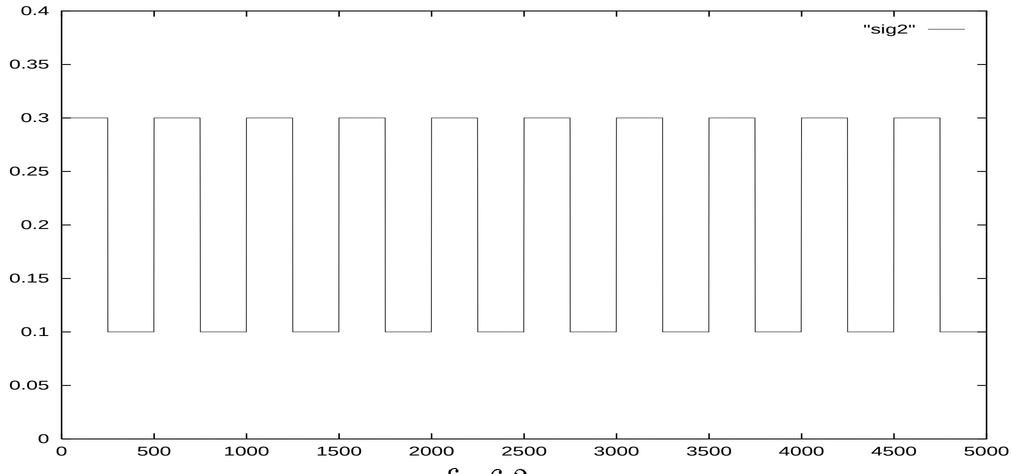

fig 6.2

Example 6.3: A continuous function

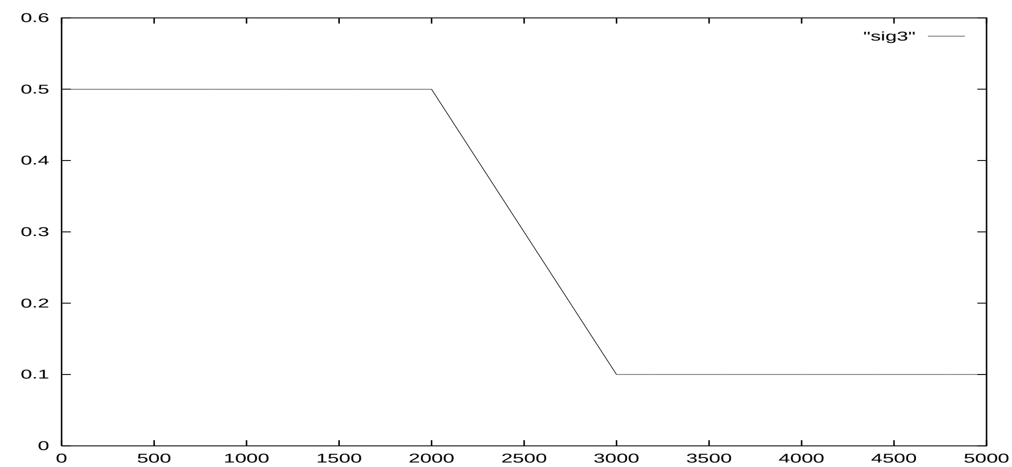

fig 6.3

6.1 Comparison on MISE:

We compute Mean Integrated Square Error as a function of window A. We plot on the same diagram the functions $MISE^{MA}(A)$, $MISE^{CMA}(A)$ and $MISE^{H}(A)$ corresponding to the Moving Average Estimator, Centred Moving Average Estimator and the Haar Basis Estimator.

Example 6.1, MISE: Few volatility jumps

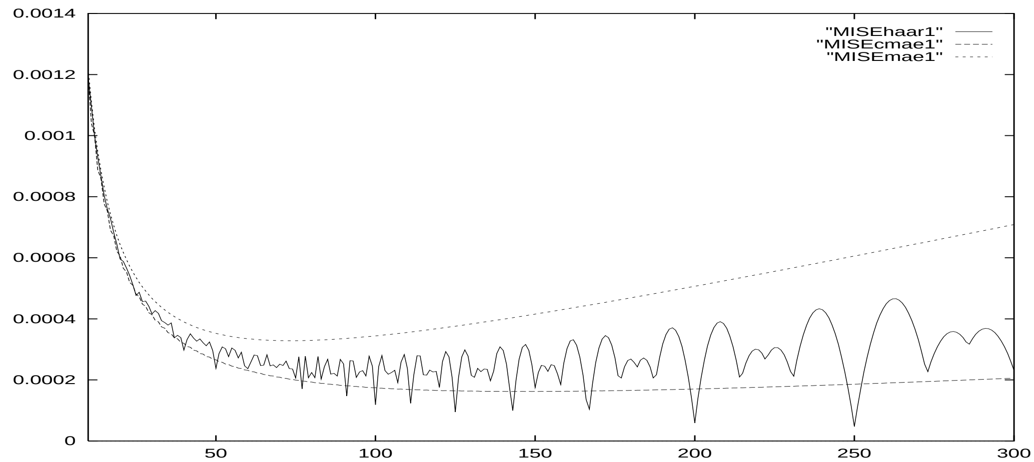

fig 6.4

Example 6.2, MISE: A lot of volatility jumps

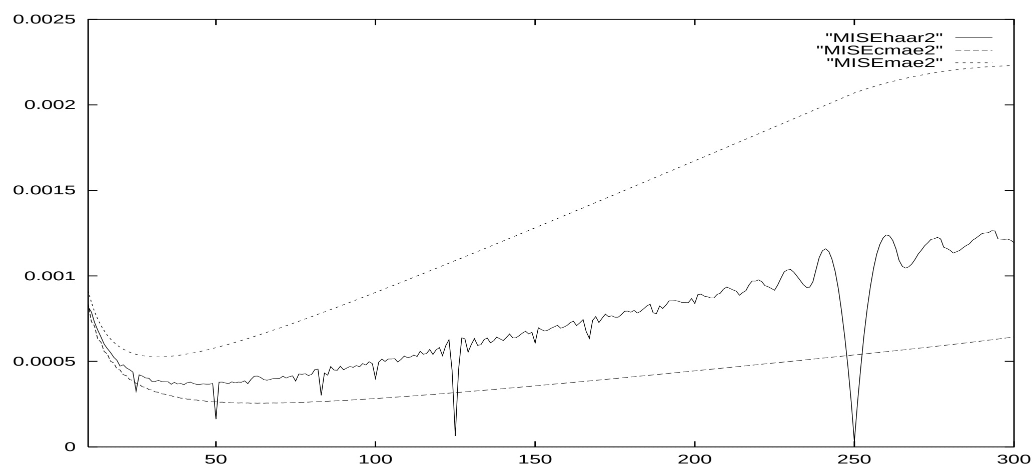

fig 6.5

Conclusion:

In every case considered, Moving Average Estimator improved by centring procedure is much more better than the classical Moving Average Estimator. As soon as there is a volatility jump, $MISE^{CMA}(A)$ is most often smaller (for Centred Moving Average Estimator) than $MISE^{H}(A)$ corresponding to the Wavelet Estimator (which remains better than MAE). Moreover, $MISE^{H}(A)$ is an oscillating function for the Haar basis Estimator when $MISE^{CMA}(A)$ is a regular function. From all these reasons, the bound of MISE is about twice for Haar basis Estimator than for CMAE. When no jump occurs, the function $MISE^{H}(A)$ is near $MISE^{CMA}(A)$.

6.2 Comparison of approximated ISE.

Since we only observe a sample path, we should consider the standard ISE. From Theorem 5.1 and Theorem 5.2 we have Central Limit Theorems for ISE, recall that:

$$ISE^{CMA}(A) = MISE^{CMA}(A) + \lambda_{CMA} A^{-1/2} \Delta^{1/2} \mathcal{N}(0,1)$$
$$ISE^{H}(A) = MISE^{H}(A) + \lambda_{H} A^{-1} \Delta^{1/2} \mathcal{N}(0,1)$$

This induces that the result on MISE is not really modified if we consider ISE. Simulations show that he functions ISE(A) are close functions MISE(A) and therefore the above conclusions on comparison of Centred Moving Average Estimator and Haar basis Estimator remain true.

6.3 Simulation of a path.

We make numerical simulation of a path for (X_t) satisfying (5), using Euler method with step $\Delta/10$. For A=110, by example, we get:

Example 6.1, Estimation of the volatility on a simulated path: Few volatility jumps

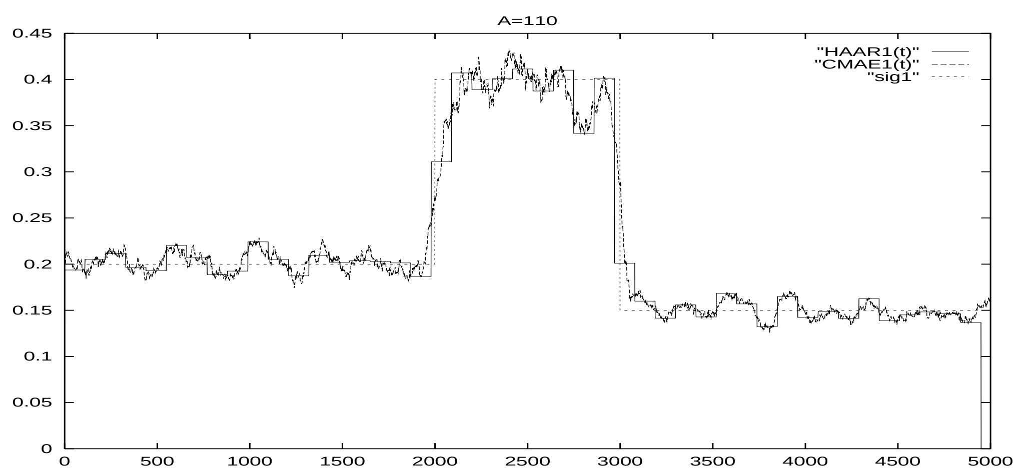

fig 6.10

Example 6.2, Estimation of the volatility on a simulated path: A lot of volatility jumps

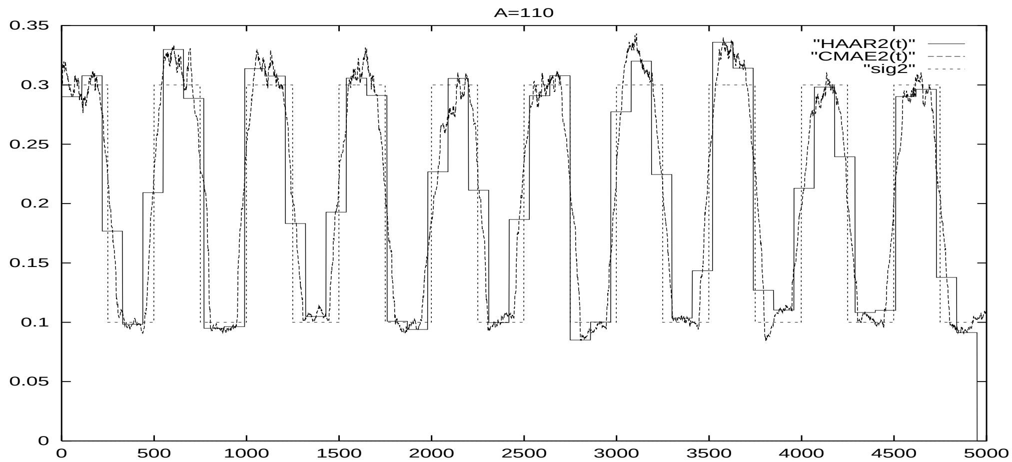

fig 6.11

Example 6.3, Estimation of the volatility on a simulated path: A continuous volatility

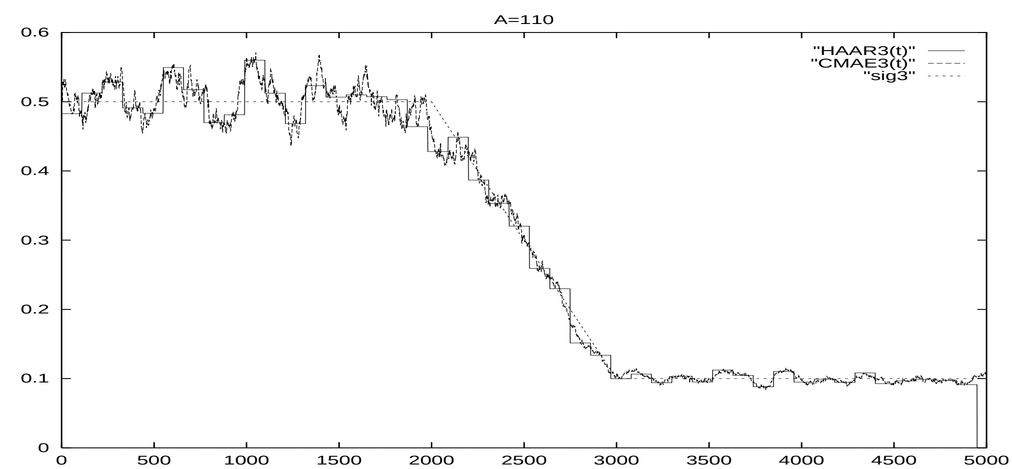

fig 6.12

On a simulated trajectory, we can compute the exact value of Integrated Square Error for each window size. We plot on the same drawing the functions $ISE^{CMA}(A)$ and $ISE^{H}(A)$ computed on the simulated path for different sizes of window and we get :

Example 6.1, Estimation of ISE on a simulated path: Few volatility jumps

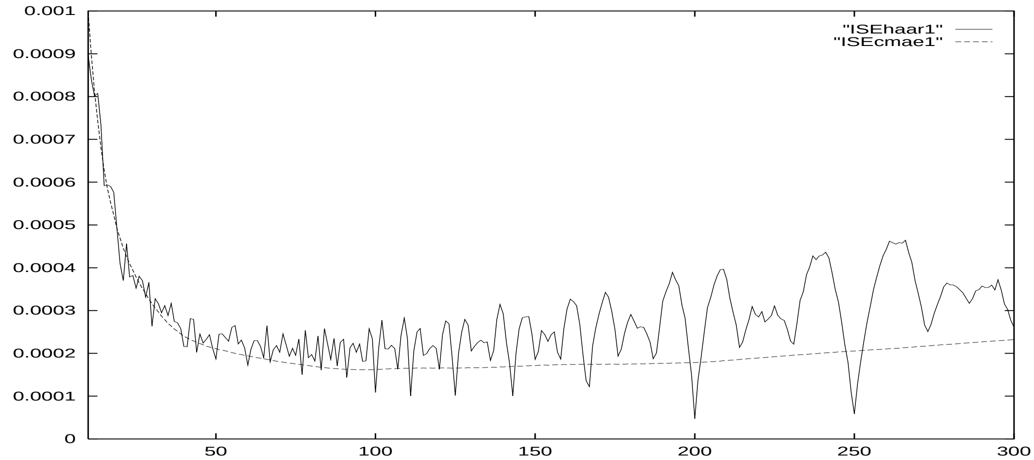

fig 6.13

Example 6.2, Estimation of ISE on a simulated path: A lot of volatility jumps

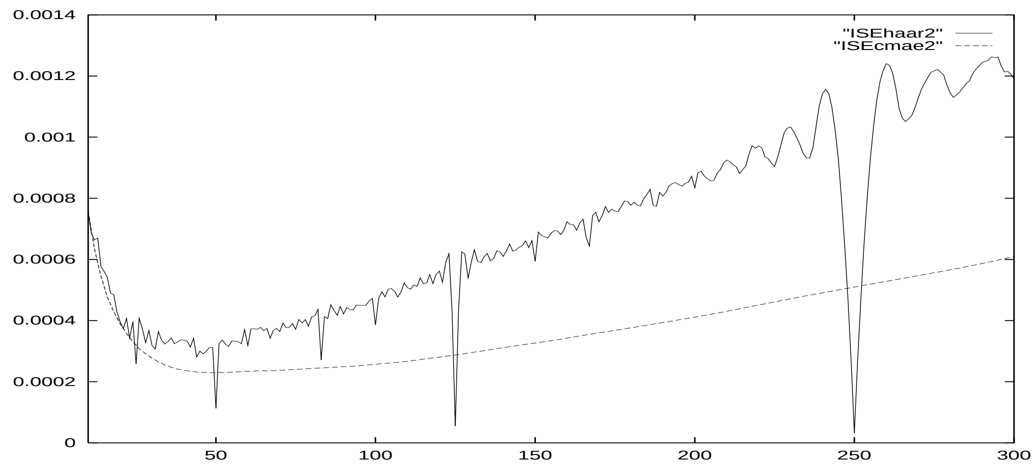

fig 6.14

Example 6.1, Estimation of ISE on a simulated path: A continuous volatility without jump

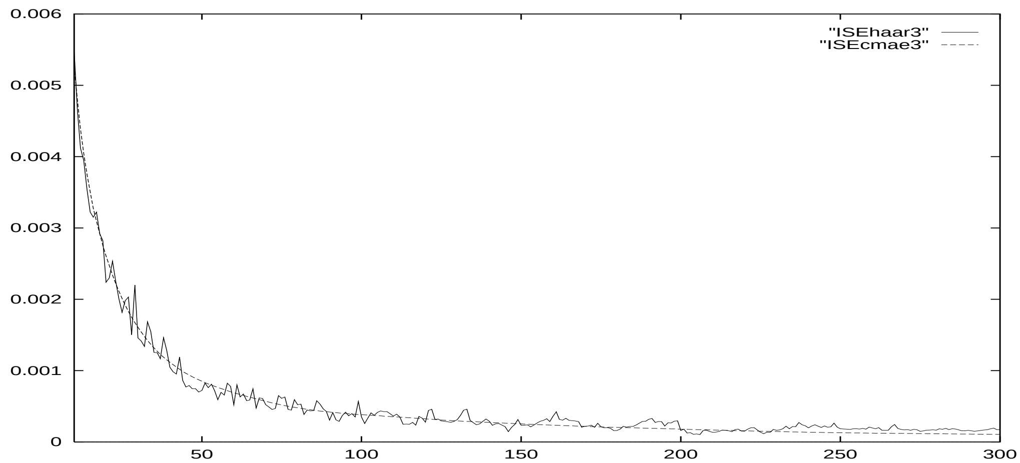

fig 6.15

All simulations give the same kind of result : as soon as there is a volatility jump ISE(A) is in most circumstances better for the Centred Moving Average Estimator than for the Haar basis Estimator. This corresponds to the above deductions.

6.4 Conclusion:

Theorical study as numerical simulations shows that for ISE and MISE, Centred Moving Average Estimator is better than Haar basis Estimator as soon as there is a volatility jump. Moreover $ISE_H(A)$ is an oscillating function of the window for of Haar basis Estimator, when $ISE_{CMA}(A)$ a regularly varying function. In this sense, Centred Moving Average Estimator is more robust with respect to the window.

This justify the use of Centred Moving Average Estimator (or MAE) to build Estimators of volatility jump times. This idea, combined with making varying size of window, will be used in our next paper.

In [1], we give applications to Estimation of volatility of real financial processes (Futures of Italian Bunds, for e.g.).

A APPENDIX (Some properties of the random variable ξ_i and bound on η_i)

First we give a bound on the second moment of the random variables η_i.

Lemma A.1 *Let X_t verify (3) and assume that (A0), (B1) are satisfied. Then $\forall i$ we have*

$$\mathbb{E}(\eta_i^2) \leq 12\sqrt{2}\Delta \left[K_T \|\mathbb{E}\theta^4(\cdot)\|_{L^\infty(t_i,t_{i+1})}\right]^{1/2} [1+\mathcal{O}(\Delta)]$$

Proof : >From (17) and Jensen inequality we have:

$$\mathbb{E}(\eta_i^2)$$

$$\leq 2\Delta^{-1}\left\{\int_{t_i}^{t_{i+1}} \mathbb{E}\left[b^4(s,X_s)\right]^{1/2} \mathbb{E}\left[(X_s - X_{t_i})^4\right]^{1/2} ds + \int_{t_i}^{t_{i+1}} \mathbb{E}\left[\theta(s)\int_{t_i}^{s} b(u,X_u)du\right]^2 ds\right\}$$

$$\leq 2\Delta^{-1}\left\{\int_{t_i}^{t_{i+1}} K_T^{1/2}\mathbb{E}\left[(X_s - X_{t_i})^4\right]^{1/2} ds + \int_{t_i}^{t_{i+1}} \left[\|\mathbb{E}\theta^4(\cdot)\|_{L^\infty}\Delta^3 \int_{t_i}^{s} \mathbb{E}b^4(u,X_u)du\right]^{1/2} ds\right\}$$

Using (42) below, we get

$$\begin{aligned}
&\mathbb{E}(\eta_i^2)\\
&\leq 2\Delta^{-1}\left\{\sqrt{2}K_T^{1/2}\Delta^2\left[36\,\|\mathbb{E}\theta^4\|_{L^\infty} + K_T\Delta^2\right]^{1/2} + K_T^{1/2}\|\mathbb{E}\theta^4\|_{L^\infty}^{1/2}\int_{t_i}^{t_{i+1}}(s-t_i)^2\,ds\right\}\\
&\leq 2\Delta K_T^{1/2}\left\{\sqrt{2}\left[36\,\|\mathbb{E}\theta^4\|_{L^\infty} + K_T\Delta^2\right]^{1/2} + \tfrac{1}{3}\Delta\|\mathbb{E}\theta^4\|_{L^\infty}^{1/2}\right\}
\end{aligned}$$

since

$$\mathbb{E}(X_s - X_{t_i})^4 \leq 8(s-t_i)^2\left[36\,\|\mathbb{E}\theta^4\|_{L^\infty(t_i,t_{i+1})} + K_T\,\Delta^2\right] \tag{42}$$

This results from the definition of the solution of a S.D.E applied to (2). After we use [25, Lemma 4.12, p. 125] and Hölder inequality and we obtain the following bounds:

$$\begin{aligned}
\mathbb{E}(X_s - X_{t_i})^4 &\leq 8\mathbb{E}\left(\int_{t_i}^{s}\theta(s)dW_s\right)^4 + 8\mathbb{E}\left(\int_{t_i}^{s} b(s,X_s)\right)^4\\
&\leq 8\left\{36\|\mathbb{E}\theta^4\|_{L^\infty(t_i,t_{i+1})} + (s-t_i)^3\int_{t_i}^{s}\mathbb{E}|b(s,X_s)|^4 ds\right\}
\end{aligned}$$

>From assumption (B1) we deduce (42) ■

We give some useful properties of the family of random variables (ξ_i) defined by (16).

Lemma A.2 *Assume that (A0) is satisfied, we have:*

$$\mathbb{E}\xi_j = 0, \forall j \qquad \textit{and} \qquad \mathbb{E}(\xi_k\xi_l) = 0 \qquad \textit{when} \quad k \neq l \tag{43}$$

$$\mathbb{E}\xi_j^2 \leq 3\|\mathbb{E}\theta^4(\cdot)\|_{L^\infty(t_j,t_{j+1})} \qquad \textit{and} \qquad \mathbb{E}\xi_{t_j}^4 \leq C_4\|\mathbb{E}\theta^8(\cdot)\|_{L^\infty(t_j,t_{j+1})} \tag{44}$$

Moreover, when $\theta(\cdot)$ is deterministic, the (ξ_j) is a family of independent random variable and we have:

$$2\xi_j = \Delta^{-1}\left[\int_j^{t_{j+1}} \theta(s)dW_s\right]^2 - \bar{\theta}_j^2 \stackrel{\mathcal{L}}{=} \bar{\theta}_j^2\left[B_1^2 - 1\right] \tag{45}$$

$$\mathbb{E}\xi_j^2 = \frac{1}{2}(\bar{\theta}_j^2)^2 \tag{46}$$

$$\mathbb{E}\xi_j^3 = (\bar{\theta}_j^2)^3; \quad \mathbb{E}\xi_j^4 = \frac{15}{4}(\bar{\theta}_j^2)^4 \tag{47}$$

More generally, $\forall m \in \mathbb{N}$, $\exists K_{2m}$ such that $\mathbb{E}\xi_j^{2m} = K_{2m}(\bar{\theta}_j^2)^m$

Proof:

From (16), (A0) and the properties of Stochastic Integral, we get (43).

The bounds (44) result from assumption (A0) and bound of Stochastic Integral Moment see e.g. [22, p. 163]. Indeed, we have for example :

$$\begin{aligned}
\Delta^4\mathbb{E}\xi_j^4 &\leq C_1\Delta\int_j^{t_{j+1}} \mathbb{E}\left[\theta(s)\int_{t_j}^{s}\theta(u)dW_u\right]^4 ds \\
&\leq C_2\Delta\int_{t_j}^{t_{j+1}} \mathbb{E}\left[\theta^8(s)\right]^{1/2}\left\{\mathbb{E}\left[\int_{t_j}^{s}\theta(u)dW_u\right]^8\right\}^{1/2} ds \\
&\leq C_3\Delta^{5/2}\int_{t_j}^{t_{j+1}} \mathbb{E}\left[\theta^8(s)\right]^{1/2}\left[\int_{t_j}^{s}\mathbb{E}\theta^8(u)du\right]^{1/2} ds \\
&\leq C_4\Delta^{-1}\int_{t_j}^{t_{j+1}} \mathbb{E}\theta^8(s)ds \\
&\leq C_4\Delta^4\|E\theta^8(.)\|_{L^\infty(t_j,t_{j+1})}
\end{aligned}$$

We turn now to the deterministic case. Let $Z_s = \int_{t_j}^{t_j+s} \theta(u)dW_u$, we have

$$dZ_s^2 = 2\,Z_s\,\theta(s)\,dW_s + \theta^2(s)\,ds$$

This gives us the first equality of (45).

We have the martingale change time :

$$\int_{t_j}^{t_{j+1}} \theta(s)dW_s = B_{(\Delta\bar{\theta}_j^2)}$$

Therefore :

$$2\xi_j = \bar{\theta}_j^2\left[(\Delta\bar{\theta}_j^2)^{-1}B^2_{(\Delta\bar{\theta}_j)^2} - 1\right] \tag{48}$$

>From the scaling property, we deduce (45). Then using (44) and the moments of gaussian random variables, we can compute all the moments of ξ_j. For $\mathbb{E}\xi_j^2$, a direct calculation is simpler ■

Remarks

* For a stochastic volatility adapted to $(\mathcal{F}_t)$, we still have (48). But $B_{(\Delta\bar{\theta}_j^2)}$ is not a Gaussian random variable and (46) is not always true as it is shown by the following counterexample. If $\theta(s) = W_s - W_{t_j}$, we get $\mathbb{E}\xi_j^2 = \frac{5}{8}\Delta^2$ and $\mathbb{E}(\bar{\theta}_j^2)^2 = \frac{7}{12}\Delta^2$, therefore $\mathbb{E}\xi_j^2 = \frac{15}{14}\mathbb{E}(\bar{\theta}_j^2)^2$.

*If $\theta^2(t)$ is bounded from below: $\forall t, \theta^2(t) \geq \nu_0 > 0$, a.s. then $\mathbb{E}\xi_j^2 \geq \frac{1}{2}\nu_0^4$.

* Lemmas A.1, A.2 and A.3 remain true when the sampling interval is not constant. Therefore, we can transpose most results of this paper to the case of variable sampling intervals $\Delta_i = t_{i+1} - t_i$ (see [1]).

Using the regularity assumption, we get a better result:

Lemma A.3 *(i) If (A0) and (A2) are satisfied. Then:*

$$\xi_j = \theta^2(t_j)\Delta^{-1}\int_{t_j}^{t_{j+1}}(W_s - W_{t_j})dW_s + \epsilon_j \tag{49}$$

with

$$\mathbb{E}\epsilon_j^2 \leq 3\Delta^{2m}\left[\mathbb{E}K^4(\omega)\right]^{1/2}\|\mathbb{E}\theta^4(\cdot)\|^{1/2}_{L^\infty(0,T)}\left[1+12\Delta\right] \tag{50}$$

Therefore

$$\mathbb{E}\xi_j^2 = \frac{1}{2}\mathbb{E}\theta^4(t_{j-k}) + \mathcal{O}(k\Delta)^m$$

(ii) If (A0) and (A1) are satisfied, then (49) holds with

$$\mathbb{E}\epsilon_j^2 \leq 12\mathbb{P}(\rho\in[t_j,t_{j+1}))^{1/2}\|\theta\;\|^{1/2}_{L^\infty(\Omega\times\mathbb{R})}\|\;\mathbb{E}\theta^4(\cdot)\;\|^{1/2}_{L^\infty(0,T)}\left[1+12\Delta\right] \tag{51}$$

and

$$\lim_{h->0}\mathbb{P}(\rho\in[t-h,t)) = 0 \tag{52}$$

Proof:

(i) Using (16), the bounds of Stochastic Integral moments and Hölder Inequality, a straightforward calculation leads to (50). Indeed we have:

$$\epsilon_j = \Delta^{-1} \int_{t_j}^{t_{j+1}} [\theta(s) - \theta(t_j)] \left[\int_{t_j}^{s} \theta(u) dW_u \right] dW_s + \Delta^{-1} \theta(t_j) \int_{t_j}^{t_{j+1}} \left[\int_{t_j}^{s} \theta(u) - \theta(t_j) dW_u \right] dW_s$$

and

$$\begin{aligned}
\frac{1}{2} \mathbb{E} \epsilon_j^2 &\leq \Delta^{-2} \mathbb{E} \left\{ \int_{t_j}^{t_{j+1}} [\theta(s) - \theta(t_j)] \left[\int_{t_j}^{s} \theta(u) dW_u \right] dW_s \right\}^2 \\
&\quad + \Delta^{-2} \mathbb{E} \left\{ \theta(t_j) \int_{t_j}^{t_{j+1}} \left[\int_{t_j}^{s} \theta(u) - \theta(t_j) dW_u \right] dW_s \right\}^2 \\
&\leq \Delta^{-2} \int_{t_j}^{t_{j+1}} \Delta^{2m} \mathbb{E} \left\{ K^2(\omega) \left[\int_{t_j}^{s} \theta(u) dW_u \right]^2 \right\} ds \\
&\quad + \Delta^{-2} \mathbb{E} \left[\theta^4(t_j) \right]^{1/2} \left\{ \mathbb{E} \left\{ \int_{t_j}^{t_{j+1}} \left[\int_{t_j}^{s} \theta(u) - \theta(t_j) dW_u \right] dW_s \right\}^4 \right\}^{1/2} \\
&\leq \Delta^{-2} \Delta^{2m} \left[\mathbb{E} K^4(\omega) \right]^{1/2} \int_{t_j}^{t_{j+1}} \left\{ \mathbb{E} \left[\int_{t_j}^{s} \theta(u) dW_u \right]^4 \right\}^{1/2} ds \\
&\quad + 2\Delta^{-2} \mathbb{E} \left[\theta^4(t_j) \right]^{1/2} \left\{ 36\Delta \int_{t_j}^{t_{j+1}} \mathbb{E} \left[\int_{t_j}^{s} \theta(u) - \theta(t_j) dW_u \right]^4 ds \right\}^{1/2} \\
&\leq 3\Delta^{2m} \left[\mathbb{E} K^4(\omega) \right]^{1/2} \| \mathbb{E} \theta^4(\cdot) \|_{L^\infty(0,T)}^{1/2} + 36\Delta^{2m} \Delta \left[\mathbb{E} K^4(\omega) \right]^{1/2} \mathbb{E} \left[\theta^4(t_j) \right]^{1/2} \\
&\leq 3\,\Delta^{2m} \left[\mathbb{E} K^4(\omega) \right]^{1/2} \| \mathbb{E} \theta^4(\cdot) \|_{L^\infty(0,T)}^{1/2} [1 + 12\Delta]
\end{aligned}$$

(ii) The proof of (51) is quite the same. We turn to (52). Since there is a finite number of volatility jump times, it suffice to prove the result with ρ_1, the first one. We have: $\mathbb{P}(\rho \in [t-h, t)) = \mathbb{E}(\varphi(h, \omega))$ with $\varphi(h, \omega) = 1$ iff $\rho_1 \in [t-h, t)), 0$ elsewhere.

But $t \wedge \rho_1$ is $\mathcal{F}_t$ adapted and also $\varphi(h, \cdot)$ which is bounded by 1.

Moreover $\lim_{h->0} \varphi(h, \omega) = 0$ a.s. From Lebesgue's Dominated Convergence Theorem, we deduce (52) ■

Remark :

We also get $\lim_{h \to 0} \mathbb{P}(\rho \in [t-h, t+h[) = \mathbb{P}(\rho = t)$ and more generally $\lim_{h \to 0} \mathbb{P}(\rho \in I_n) = \mathbb{P}(\rho = t)$ when I_n is an interval such that $t \in I_n$ and $\mathrm{diam}(I_n) = h \to 0$.

B Verification of Lyapunov Condition in Central Limit Theorem for Integrated Square Error.

End of Proof of Th.4.1 (CLT for ISE for the Haar basis Estimator)
Lyapunov Condition is

$$\lim_{\Delta\to 0}(A\Delta)^4 U_n^{-4}\sum_{k=0}^{N/A-1} I\!E\left[\zeta_k^2 - I\!E\zeta_k^2\right]^4 = 0$$

This is equivalent to

$$\lim_{\Delta\to 0}\Delta^2\sum_{k=0}^{N/A-1} A^8 I\!E\left[\zeta_k^2 - I\!E\zeta_k^2\right]^4 = 0$$

We compute this in function of the moments of $A^2\zeta_k^2$. We denote them:

$$M_1 = A^2\, I\!E\zeta_k^2, \qquad M_2 = A^4\, I\!E\zeta_k^4, \qquad M_3 = A^6\, I\!E\zeta_k^6, \qquad M_4 = A^8\, I\!E\zeta_k^8,$$

We have:

$$A^8 I\!E\left[\zeta_k^2 - I\!E\zeta_k^2\right]^4 = M_4 - 3M_1^4 - 4M_1\,M_3 + 6M_2\,M_1^2 \tag{53}$$

We turn now to the moments $M_1, M_2\ldots$ Since $\theta(.)$ is deterministic, Lemma A.2 gives us $I\!E(\xi_j^{2m}) = K_{2m}(I\!E\xi_j^2)^m$. We deduce:

$$\begin{aligned}
M_1 &= \sum_{j=kA}^{(k+1)A-1} I\!E\xi_j^2\\
M_2 &= M_1^2 + (K_4-1)\sum_{j=kA}^{(k+1)A-1}(I\!E\xi_j^2)^2\\
M_3 &= M_1^3 + 3(K_4-1)\,M_1\sum_{j=kA}^{(k+1)A-1} I\!E(\xi_{j_1}^2)^2 + (K_6-3K_4+2)\sum_{j=kA}^{(k+1)A-1} I\!E(\xi_{j_1}^2)^3\\
M_4 &= M_1^4 + 6(K_4-1)\,M_1^2\sum_{j=kA}^{(k+1)A-1} I\!E(\xi_j^2)^2 + 4(K_6-3K_4+2)\,M_1\sum_{j=kA}^{(k+1)A-1} I\!E(\xi_j^2)^3\\
&\quad +(K_8-1)\sum_{j=kA}^{(k+1)A-1} I\!E(\xi_j^2)^4
\end{aligned}$$

Indeed

$$M_4 = \sum_{j_1,j_2,j_3,j_4} I\!E(\xi_{j_1}^2\,\xi_{j_2}^2\,\xi_{j_3}^2\,\xi_{j_4}^2)$$

$$
\begin{aligned}
&= \sum_{j_1 \neq j_2 \neq j_3 \neq j_4} \mathbb{E}(\xi_{j_1}^2)\mathbb{E}(\xi_{j_2}^2)\mathbb{E}(\xi_{j_3}^2)\mathbb{E}(\xi_{j_4}^2) + 6 \sum_{j_1 \neq j_2, j_3} \mathbb{E}(\xi_{j_1}^4)\mathbb{E}(\xi_{j_2}^2)\mathbb{E}(\xi_{j_3}^2) \\
&\quad +4 \sum_{j_1 \neq j_2} \mathbb{E}(\xi_{j_1}^6)\mathbb{E}(\xi_{j_2}^2) + \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_j^8) \\
&= M_1^4 + 6(K_4 - 1) \sum_{j_1 \neq j_2, j_3} \mathbb{E}(\xi_{j_1}^2)^2 \mathbb{E}(\xi_{j_2}^2)\mathbb{E}(\xi_{j_3}^2) \\
&\quad +4(K_6 - 1) \sum_{j_1 \neq j_2} \mathbb{E}(\xi_{j_1}^2)^3 \mathbb{E}(\xi_{j_2}^2) + (K_8 - 1) \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_j^2)^4 \\
&= M_1^4 + 6(K_4 - 1)M_1^2 \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_{j_1}^2)^2 \\
&\quad +(4K_6 - 12K_4 + 8) \sum_{j_2, j_3} E(\xi_{j_3}^2)^3 \mathbb{E}(\xi_{j_2}^2) + (K_8 - 1) \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_j^2)^4 \\
&= M_1^4 + 6(K_4 - 1)M_1^2 \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_{j_1}^2)^2 + 4(K_6 - 3K_4 + 2)M_1 \sum_{j_2, j_3} \mathbb{E}(\xi_{j_3}^2)^3 \\
&\quad +(K_8 - 1) \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_j^2)^4
\end{aligned}
$$

and

$$
\begin{aligned}
M_3 &= \sum_{j_1, j_2, j_3} \mathbb{E}(\xi_{j_1}^2 \xi_{j_2}^2 \xi_{j_3}^2) \\
&= \sum_{j_1 \neq j_2 \neq j_3} \mathbb{E}(\xi_{j_1}^2)\mathbb{E}(\xi_{j_2}^2)\mathbb{E}(\xi_{j_3}^2) + 3 \sum_{j_1 \neq j_2} \mathbb{E}(\xi_{j_1}^4)\mathbb{E}(\xi_{j_2}^2) + \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_{j_1}^6) \\
&= M_1^3 + 3(K_4 - 1) \sum_{j_1 \neq j_2} \mathbb{E}(\xi_{j_1}^2)^2 \mathbb{E}(\xi_{j_2}^2) + (K_6 - 1) \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_{j_1}^2)^3 \\
&= M_1^3 + 3(K_4 - 1)M_1 \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_{j_1}^2)^2 + (K_6 - 3K_4 + 2) \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_{j_1}^2)^3
\end{aligned}
$$

Let

$$
\begin{aligned}
T_2 &= (K_4 - 1) \sum_{j=kA}^{(k+1)A-1} (\mathbb{E}\xi_j^2)^2 \\
T_3 &= (K_6 - 3K_4 + 2) \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_{j1}^2)^3 \\
T_4 &= (K_8 - 1) \sum_{j=kA}^{(k+1)A-1} \mathbb{E}(\xi_j^2)^4
\end{aligned}
$$

we get:

$$\begin{aligned} M_2 &= M_1^2 + T_2 \\ M_3 &= M_1^3 + 3M_1T_2 + T_3 \\ M_4 &= M_1^4 + 6M_1^2T_2 + 4M_1T_2 + T_4 \end{aligned}$$

Therefore

$$M_4 - 3M_1^4 - 4M_1M_3 + 6M_2M_1^2 = T_4$$

combined with (53), this induces

$$\begin{aligned} \Delta^2 \sum_{k=0}^{N/A-1} A^8 \mathbb{E}\left[\zeta_k^2 - \mathbb{E}\zeta_k^2\right]^4 &= \Delta(K_8 - 1) \sum_{j=0}^{N-1} \Delta \mathbb{E}(\xi_j^2)^4 \\ &= \Delta(K_8 - 1) \sum_{j=0}^{N-1} \Delta(\bar{\theta}_j^2)^4 \end{aligned}$$

■

End of Proof of Th.4.2 (CLT for ISE for CMAE)
Verification of Lyapunov Condition: $L_y \to 0$ in Probability, where

$$L_y = \sum_{i=0}^{N-1} \mathbb{E}\left[|\ A^{-2}\tilde{X}_{N,i}\ |^3|\ \mathcal{F}_i\right] = Cte\,\Delta^{3/2}A^{-9/2} \sum_{i=0}^{N-1} \mathbb{E}\left[|\ X_{N,i}\ |^3|\ \mathcal{F}_i\right]$$

It suffices to prove convergence in $L^2(\Omega)$. But we have:

$$\begin{aligned} \|L_y\|_{L^2(\Omega)} &\le Cte\,\Delta^{3/2}A^{-9/2} \sum_{i=0}^{N-1} \|\mathbb{E}\left[|\ X_{N,i}\ |^3|\ \mathcal{F}_i\right]\|_{L^2(\Omega)} \\ &\le Cte\,\Delta^{3/2}A^{-9/2} \sum_{i=0}^{N-1} \|X_{N,i}^3\|_{L^2(\Omega)} \\ &\le Cte\,\Delta^{1/2} T \|\theta\|^3_{L^\infty(O,T)} \to 0 \qquad as\ \Delta \to 0 \end{aligned}$$

Indeed, since

$$X_{N,i} = \beta(i,i)\nu_i + 2A\,\xi_i\upsilon_i$$

with $\nu_i = [\xi_i^2 - \mathbb{E}\xi_i^2]$ and $\upsilon_i = \sum_{j<i} A^{-1}\beta(i,j)\xi_j$.
We use standard inequalities to deduce:

$$\|X_{N,i}^3\|_{L^2(\Omega)} \tag{54}$$

$$\le C\,A^3\left[\|\nu_i^3\|_{L^2(\Omega)} + \|\nu_i^2\xi_i\upsilon_i\|_{L^2(\Omega)} + \|\nu_i\xi_i^2\upsilon_i^2\|_{L^2(\Omega)} + \|\xi_i^3\upsilon_i^3\|_{L^2(\Omega)}\right] \tag{55}$$

$$\leq C A^3 \left[\mathbb{E}(\nu_i^6)^{1/2} + \mathbb{E}(\nu_i^4 \xi_i^2)^{1/2} \mathbb{E}(v_i^2)^{1/2} + \mathbb{E}(\nu_i^2 \xi_i^4)^{1/2} \mathbb{E}(v_i^4)^{1/2} + \mathbb{E}(\xi_i^6) \mathbb{E}(v_i^6)^{1/2} \right] \quad (56)$$

$$\leq C\, A^{9/2} \left[1 + A^{-1/2} + A^{-1} + A^{-3/2} \right] \|\theta^{12}\|_{L^\infty(O,T)} \quad (57)$$

The last inequality results from the indepedency of ξ_i and v_i and the following bounds :

$$\begin{aligned} \mathbb{E}(v_i^6) &= \sum_{j_1,j_2,j_3} \mathbb{E}(\xi_{j_1}^2 \xi_{j_2}^2 \xi_{j_3}^2) \leq A^3 \|\theta^{12}\|_{L^\infty(O,T)} \\ \mathbb{E}(v_i^4) &\leq A^2 \|\theta^8\|_{L^\infty(O,T)} \\ \mathbb{E}(v_i^2) &\leq A \|\theta^4\|_{L^\infty(O,T)} \end{aligned}$$

which are deduced from indepedency of (ξ_i), (38) and Lemma A.2 ∎

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